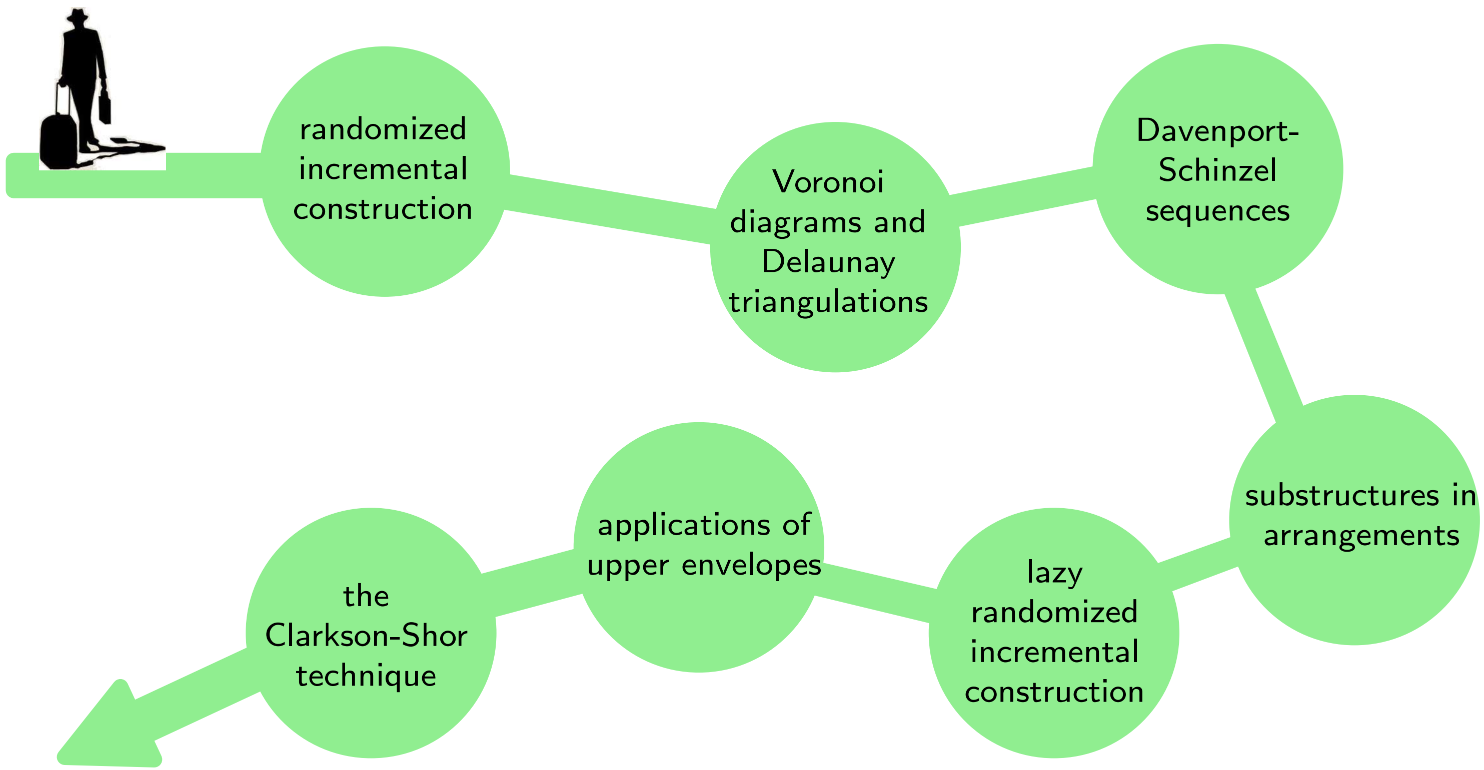
COMPUTATIONAL GEOMETRY

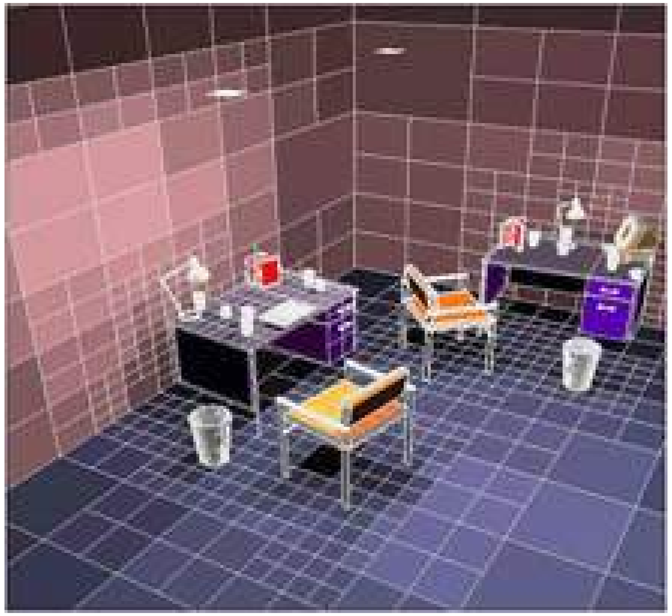
An Introduction Through Randomized Incremental Algorithms

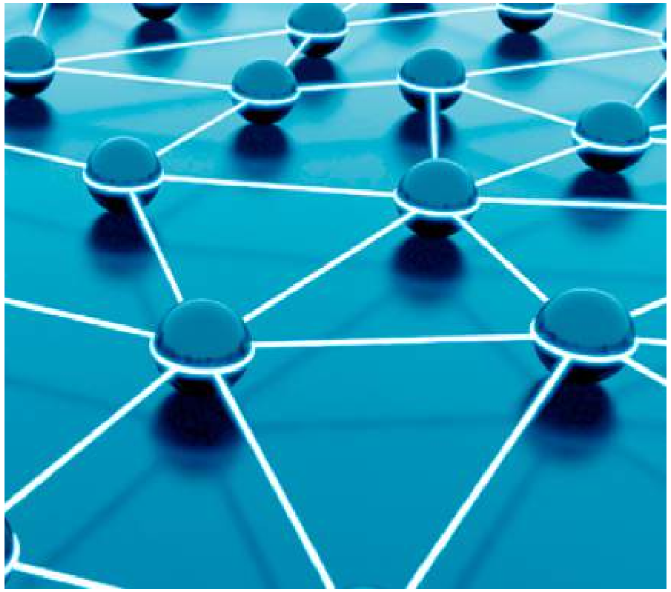


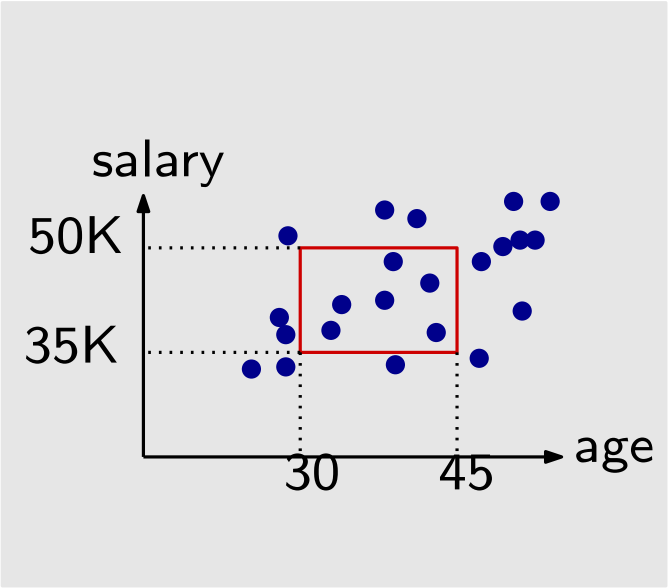
Mark de Berg (TU Eindhoven)

Geometry is everywhere . . .

- geographic information systems
- computer-aided design and manufacturing
- virtual reality
- robotics
- computational biology
- sensor networks
- databases
- and more . . .





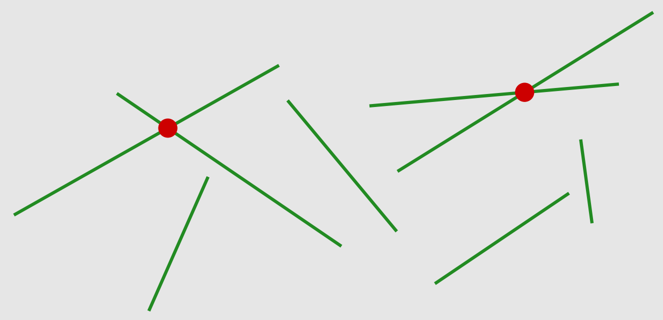


Computational Geometry

area within algorithms research dealing with spatial data

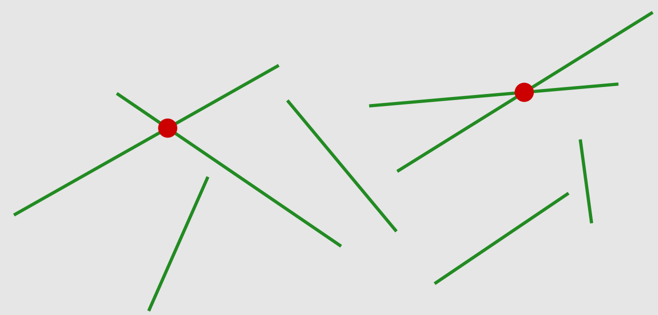
- aim for provably correct solutions (no heuristics)
- theoretical analysis of running time, memory usage: $O(\cdots)$

example problem: line-segment intersection



Compute all k intersections in a set S of n line segments.

example problem: line-segment intersection



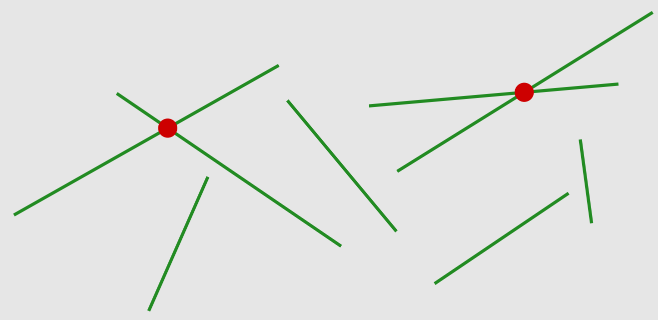
Compute all k intersections in a set S of n line segments.

1. **for** every pair of segments in S
2. **do** compute (possible) intersection

- running time $O(n^2)$
- can we do better if k is small?
  yes: $O(n \log n)$

Computational Geometry

example problem: line-segment intersection



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Computational geometry

- focus on scale-up behavior
- basic operations are assumed available (compute intersection of two lines, distance between two points, etc.)

Computational Geometry: Tools of the Trade

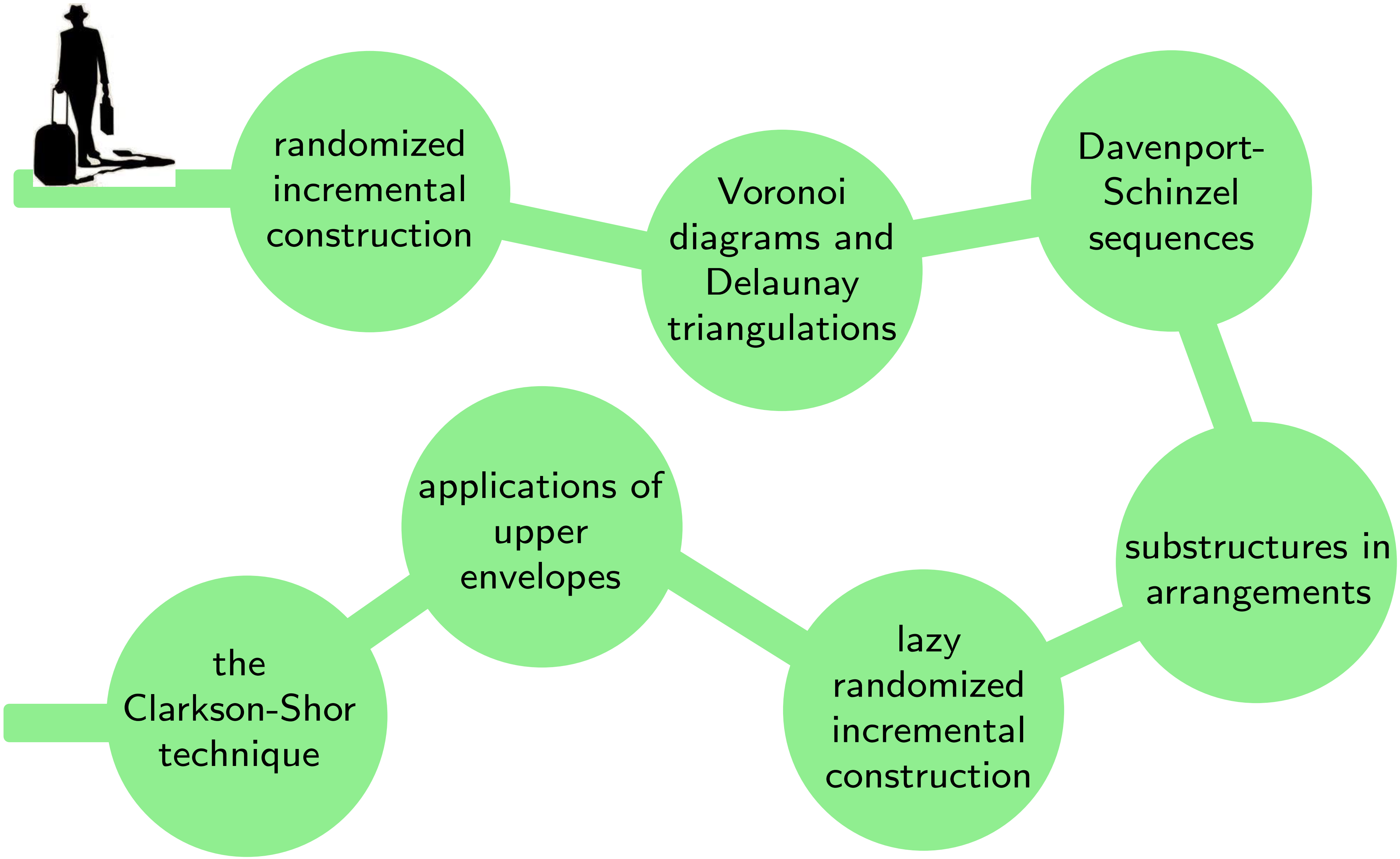
Algorithmic design techniques and tools

- plane sweep
- geometric divide-and-conquer
- randomized incremental construction
- parametric search
- (multi-level) geometric data structures

Geometric structures and concepts

- Voronoi diagrams and Delaunay triangulations
- arrangements
- cuttings, simplicial partitions, polynomial partitions
- coresets

Course Overview



Course Overview

randomized incremental construction

Voronoi diagrams and Delaunay triangulations

Davenport-Schinzel sequences

substructures in arrangements

lazy randomized incremental construction

applications of upper envelopes

the Clarkson-Shor technique

Warm-up Exercise

Warm-up Exercise

Analyze worst-case and the expected running time of the following algorithm

$\textsc{ParanoidMax}(A)$

$\triangleright$ computes maximum in an array $A[0..n-1]$

Randomly permutate the elements in the array A

$max \leftarrow A[0]$

for $i \leftarrow 1$ **to** $n-1$ **do**

  if $A[i] > max$ **then**

    $max \leftarrow A[i]$

    to be on the safe side, check if $A[i]$ is indeed the largest element in $A[0..i]$

return max

Warm-up Exercise

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Randomly permutate the elements in the array A
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for $i \leftarrow 1$ **to** $n-1$ **do**
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        $max \leftarrow A[i]$
        for $j \leftarrow 0$ **to** $i-1$ **do**
            if $A[j] > max$ **then** error
return max

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      if $A[j] > max$ **then** error

return max

- generates permutation uniformly at random
- assume this can be done in $O(n)$ time

Worst-case analysis

$$\begin{aligned}\text{running time} &= O(n) + \textstyle\sum_{i=1}^{n-1} (\text{worst-case time for } i\text{-th iteration}) \\ &= O(n) + \textstyle\sum_{i=1}^{n-1} O(i) \\ &= O(n^2)\end{aligned}$$

Analysis of expected running time

$$
\begin{aligned}
\mathrm{E}\left[\text{running time}\right] &= \mathrm{E}\left[O(n) + \textstyle\sum_{i=1}^{n-1} \text{time for } i\text{-th iteration}\right] \\
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\mathrm{E}\left[\text{time for } i\text{-th iteration}\right] &= \Pr\left[\text{max changes in } i\text{-th iteration}\right] \cdot O(i) \\
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$$\mathrm{E}\left[\text{running time}\right] = \mathrm{E}\left[O(n) + \textstyle\sum_{i=1}^{n-1} \text{time for } i\text{-th iteration}\right]$$

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backwards analysis

max changes when adding $A[i]$ to $\{A[0], \ldots, A[i-1]\}$ $\iff$ max changes when removing $A[i]$ from $\{A[0], \ldots, A[i]\}$

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with respect to random choices of algorithm, no assumptions on input distribution

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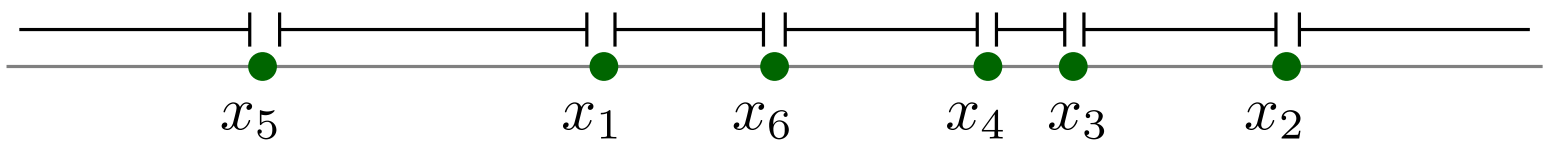
Sorting using (Randomized) Incremental Construction

A geometric view of sorting



Input: A set $S = \{x_1, \dots, x_n\}$ of n points in $\mathbb{R}^1$
Output: Sorted set $\mathcal{I}$ of intervals into which S partitions $\mathbb{R}^1$

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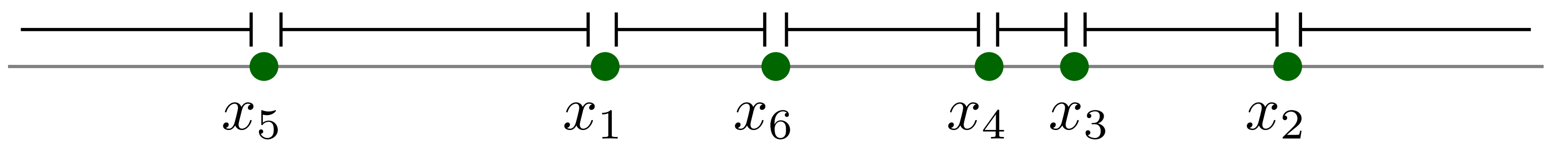

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Incremental construction:

Add points one by one, and update $\mathcal{I}$ after each addition

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IC-SORT(S)

$\mathcal{I} \leftarrow \{[-\infty, +\infty]\}$
for $j \leftarrow 1$ **to** n **do**
    Find interval $I = [x, x']$ in $\mathcal{I}$ that contains x_j
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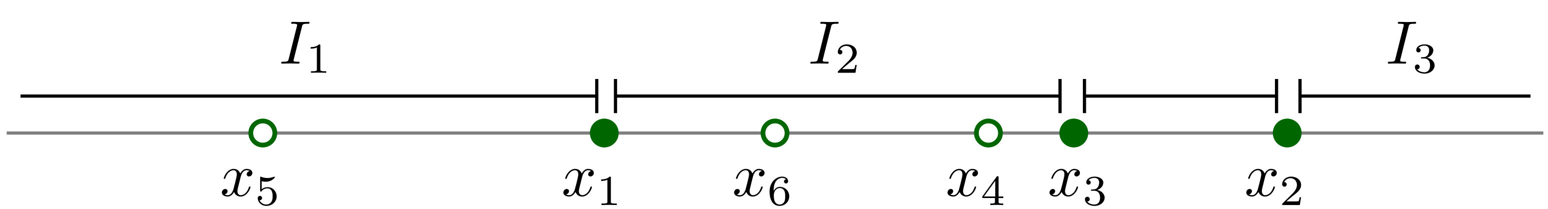

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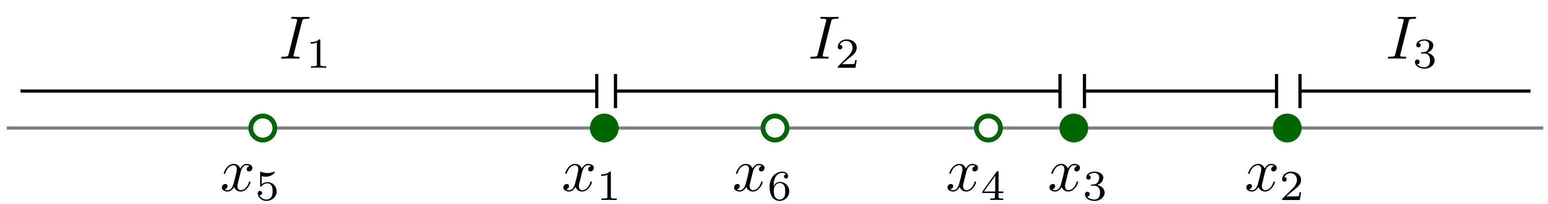

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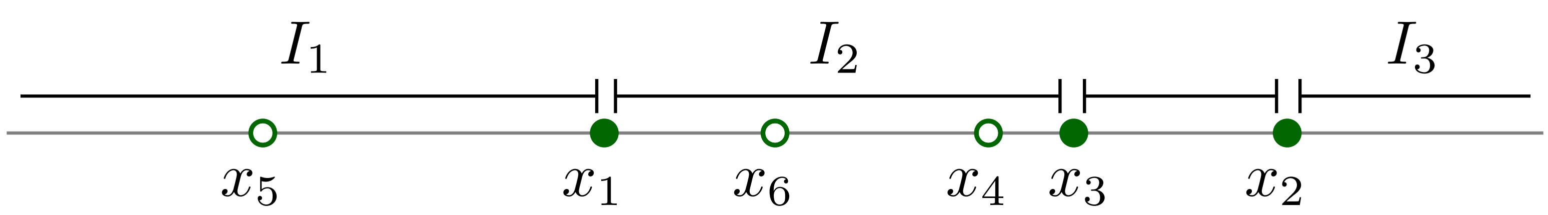

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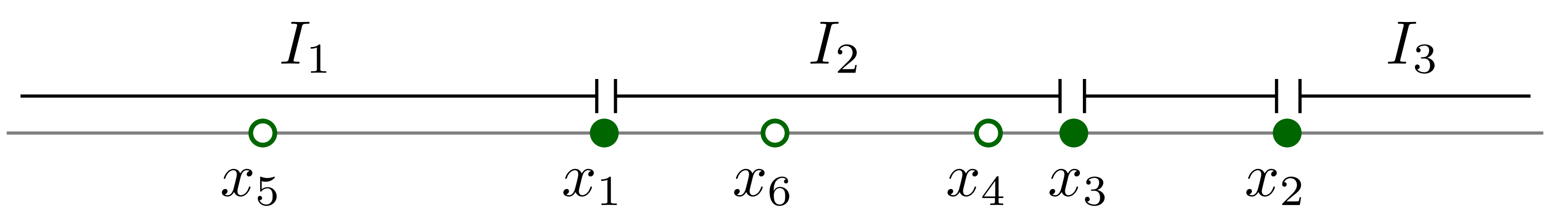

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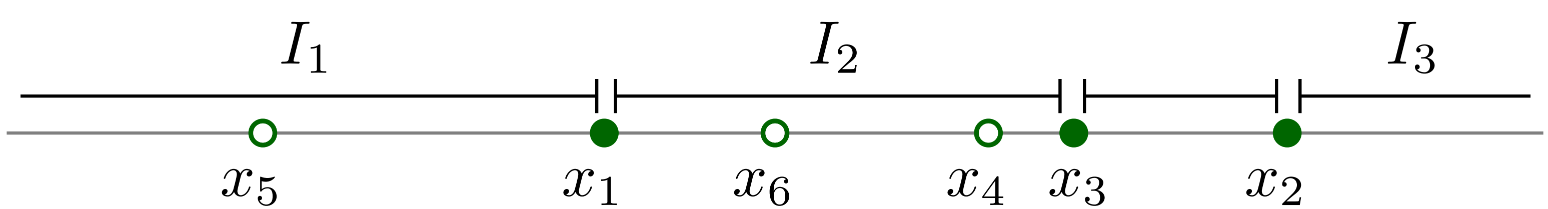

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IC-SORT(S) Put points x_i in random order

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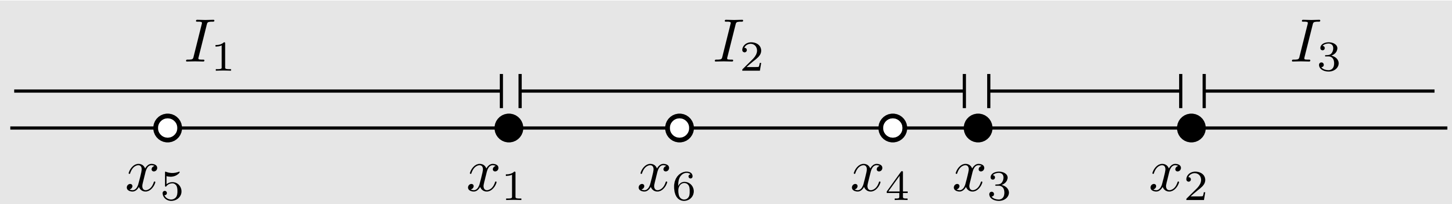
- expected:



apply backwards analysis

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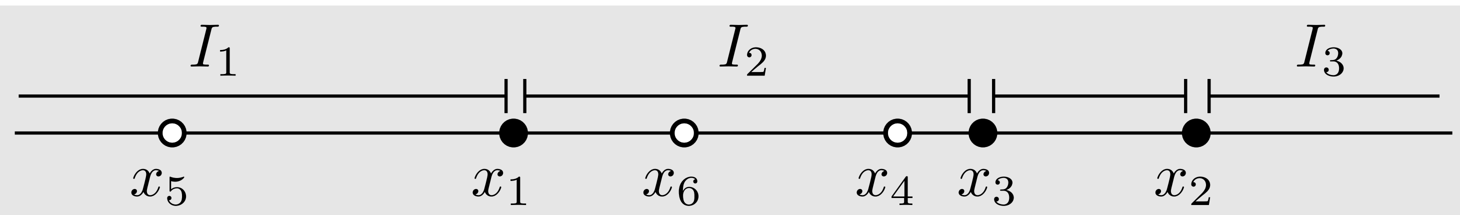
at most
$1 + (n - j + 1) \cdot \frac{2}{j}$



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$$\sum_{j=1}^{n} \left(1 + \frac{2(n-j+1)}{j}\right) = O\left(n + n\sum_{j=1}^{n} \frac{1}{j}\right) = O(n \log n)$$

Running time: $O(\sum_{j=1}^{n}$ (size of conflict list split in j-th iteration))

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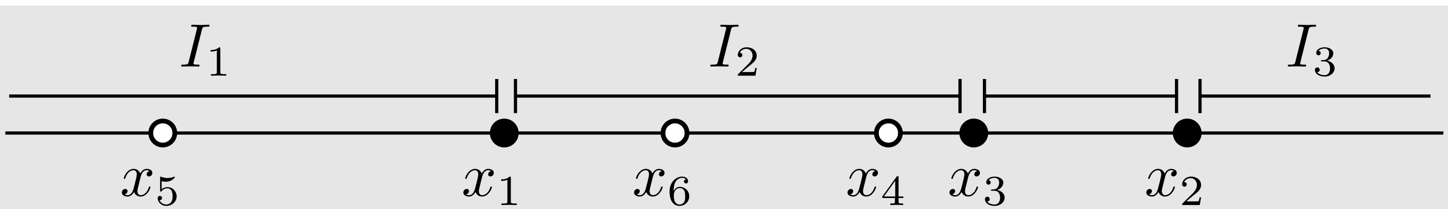

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Randomized Incremental Construction: The Framework

- S = set of n input objects
- $\mathcal{C}(S)$ = set of configurations defined by S
  - $D(\Delta) \subset S$ = defining set of $\Delta \in \mathcal{C}(S)$
    size should be bounded by a fixed constant
  - $K(\Delta) \subset S$ = conflict list of $\Delta \in \mathcal{C}(S)$
    $K(\Delta) \cap D(\Delta) = \emptyset$ for all Δ

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    $K(\Delta) \cap D(\Delta) = \emptyset$ for all Δ

For $S' \subseteq S$, define $\mathcal{C}_{\mathrm{act}}(S') = \{\Delta \in \mathcal{C}(S) : D(\Delta) \subseteq S'$ and $K(\Delta) \cap S' = \emptyset\}$
to be the set of configurations that are active with respect to S'

Goal: compute set $\mathcal{C}_{\mathrm{act}}(S)$ of active configurations with respect to S

Randomized Incremental Construction: The Framework

- S = set of n input objects
- $\mathcal{C}(S)$ = set of configurations defined by S
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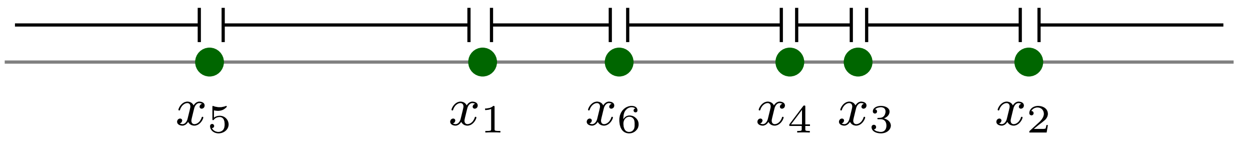
Goal: compute set $\mathcal{C}_{\text{act}}(S)$ of active configurations with respect to S

Example: sorting



$\mathcal{C}(S) := \{ \ [x_i, x_j] : x_i, x_j \in S \cup \{-\infty, +\infty\} \text{ and } x_i < x_j \ \}$

Randomized Incremental Construction: The Algorithm

```
RIC-ALGORITHM(S)
Compute a random permutation x_1, ..., x_n of the objects in S
C_act ← {active configurations with respect to ∅}
Intitialize conflict lists of configurations Δ ∈ C_act
for j ← 1 to n do
    Remove configurations from C_act that are in conflict with x_j
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    Construct conflict lists of new active configurations
return C_act
```

Randomized Incremental Construction: The Algorithm

RIC-ALGORITHM(S)

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    Determine new active configurations and insert them into C_act
    Construct conflict lists of new active configurations
return C_act
```

To find configurations that become inactive:

- for each x_j maintain a list of all configurations $\Delta \in \mathcal{C}_{\text{act}}$ with $x_j \in K(\Delta)$
- for each configuration $\Delta \in \mathcal{C}_{\text{act}}$ maintain its conflict list $K(\Delta)$

Randomized Incremental Construction: The Algorithm

RIC-ALGORITHM(S)

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Randomized Incremental Construction: The Algorithm

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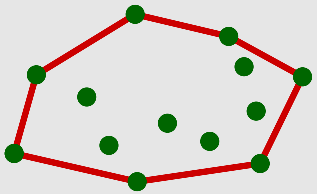
Theorem. Let $S_j := \{x_1, \ldots, x_j\}$. Then

(i) $\mathrm{E}\left[\, |\mathcal{C}_{\mathrm{act}}(S_j) \setminus \mathcal{C}_{\mathrm{act}}(S_{j-1})| \,\right] = O\left(\frac{\mathrm{E}[\text{size of } \mathcal{C}_{\mathrm{act}}(S_j)]}{j}\right)$

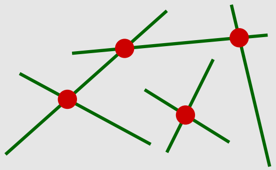
(ii) The total size of the conflict lists of the active configurations appearing over the course of the algorithm is $O\left(\sum_{j=1}^{n} \frac{n}{j^2} \cdot \mathrm{E}\left[\, |\mathcal{C}_{\mathrm{act}}(S_j)| \,\right]\right)$

Exercises

1. Give an algorithm that computes (all edges of) the convex hull of a set S of n points in the plane that runs in $O(n \log n)$ expected time.



2. Give an algorithm that computes all k intersections in a set S of n segments in the plane that runs in $O(n \log n + k)$ expected time.



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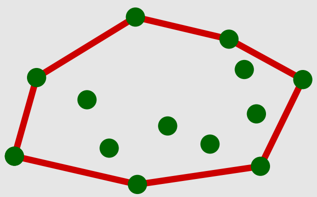

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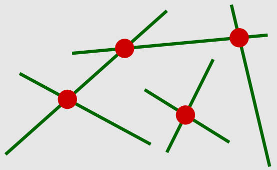

The framework

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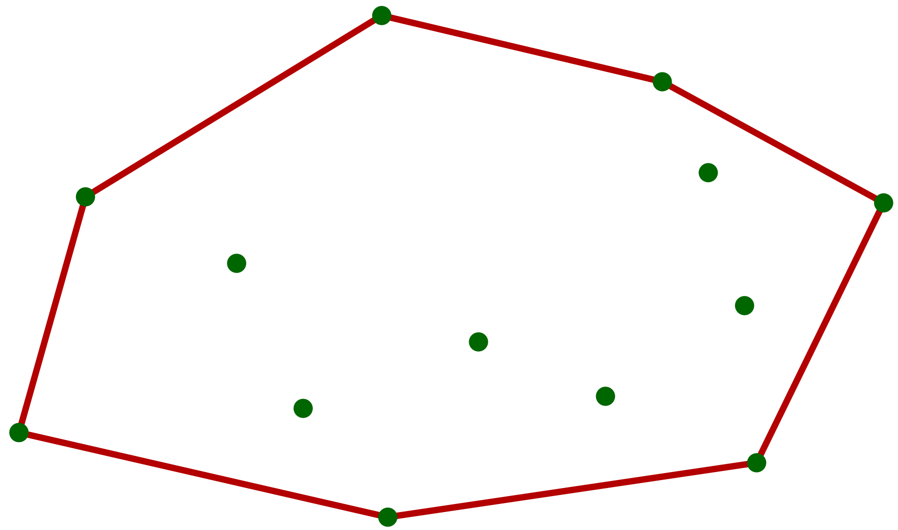

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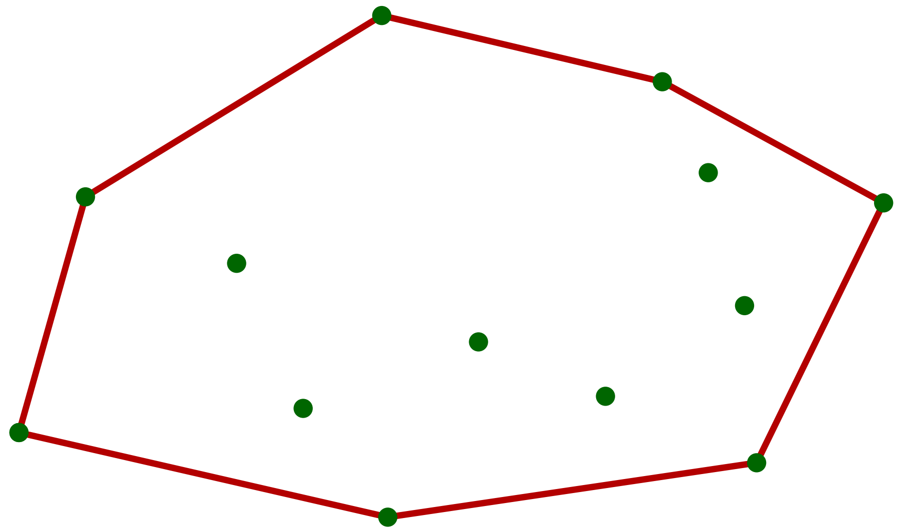

The framework

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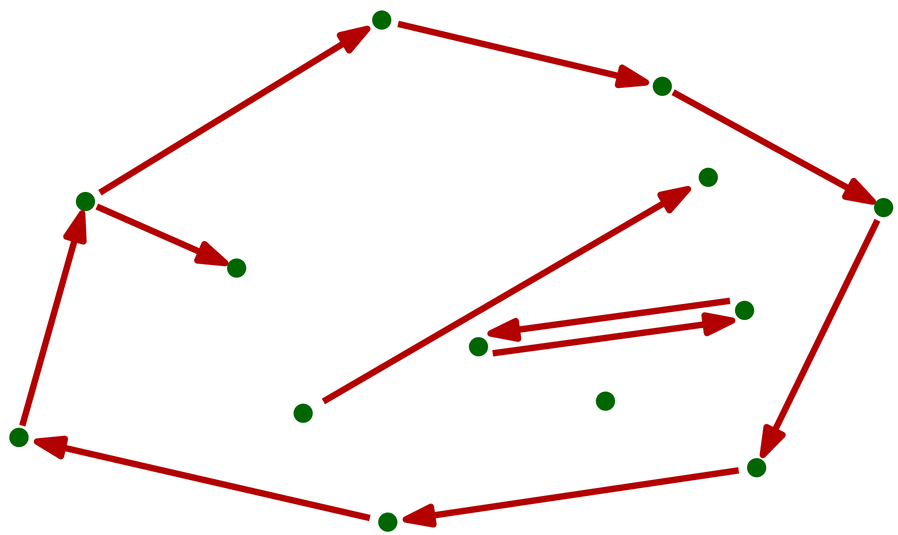

The framework

- S = set of n input objects   the points
- $\mathcal{C}(S)$ = set of configurations defined by S directed segments
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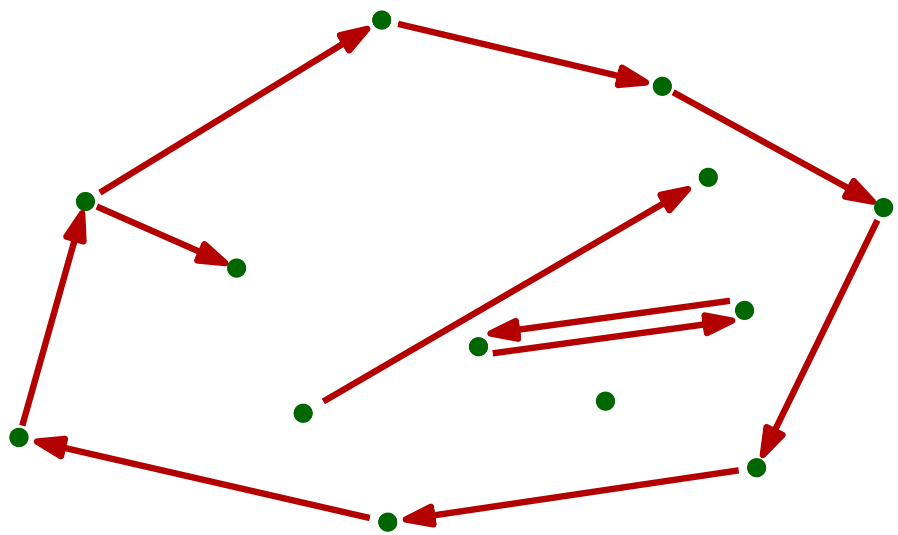

The framework

- S = set of n input objects   the points
- $\mathcal{C}(S)$ = set of configurations defined by S directed segments
  - $D(\Delta) \subset S$ = defining set of $\Delta \in \mathcal{C}(S)$ endpoints
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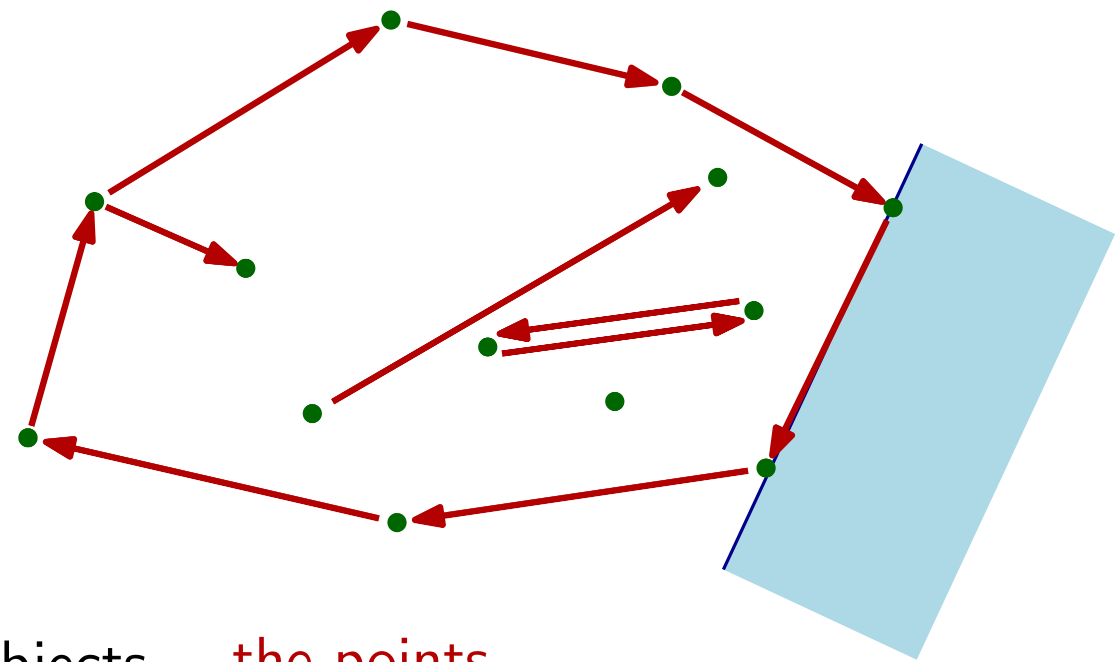

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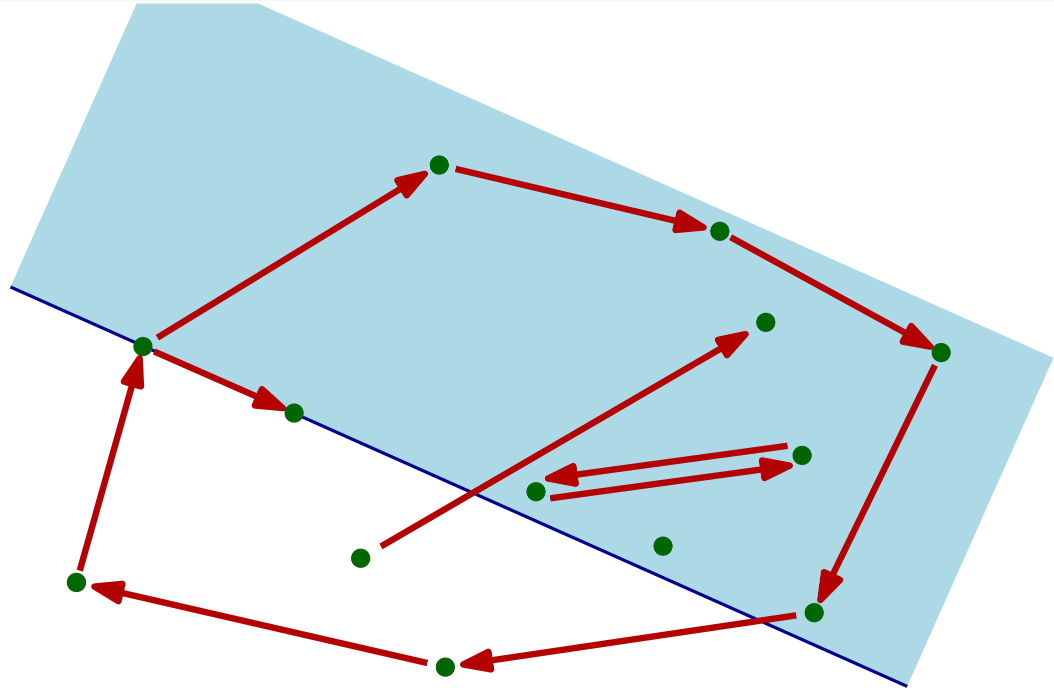

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Randomized Incremental Construction: The Algorithm

RIC-ALGORITHM(S)

Compute a random permutation $x_1, \ldots, x_n$ of the objects in S
$\mathcal{C}_{\text{act}} \leftarrow$ {active configurations with respect to $\emptyset$}
Intitialize conflict lists of configurations $\Delta \in \mathcal{C}_{\text{act}}$
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    Remove configurations from $\mathcal{C}_{\text{act}}$ that are in conflict with x_j
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return $\mathcal{C}_{\text{act}}$

Randomized Incremental Construction: The Algorithm

RIC-Algorithm(S)

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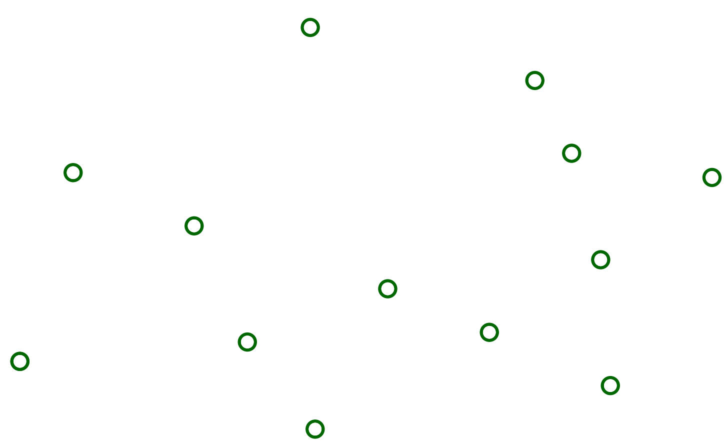

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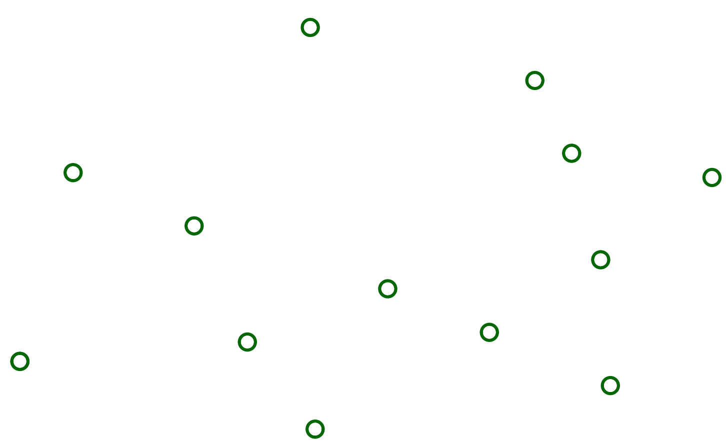

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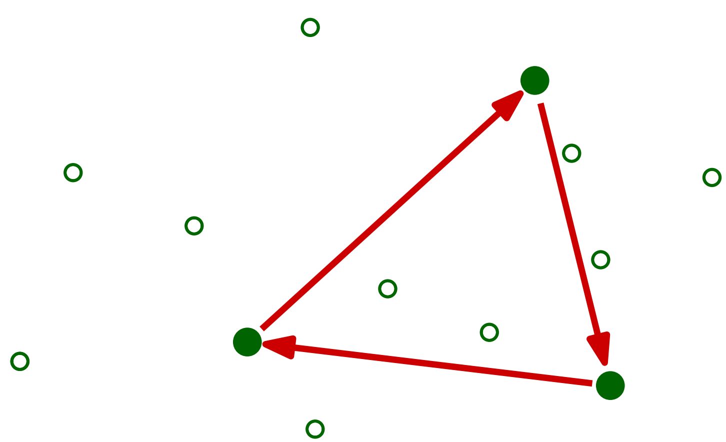

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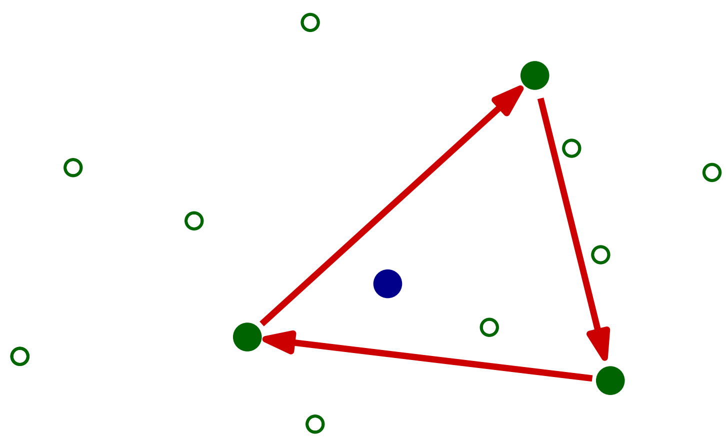

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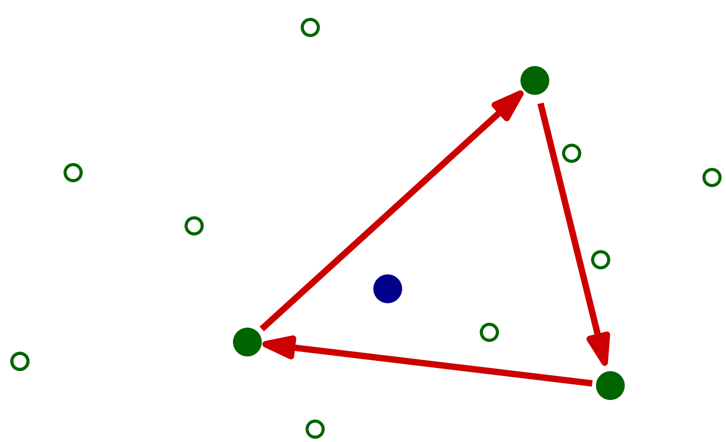

- not in conflict with active configs
- no new active configs

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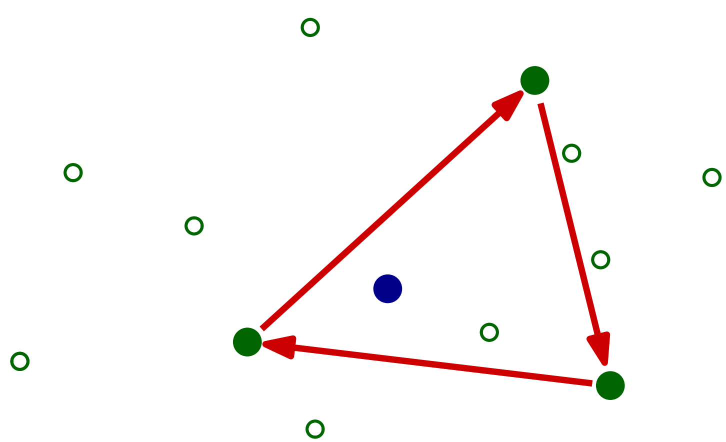

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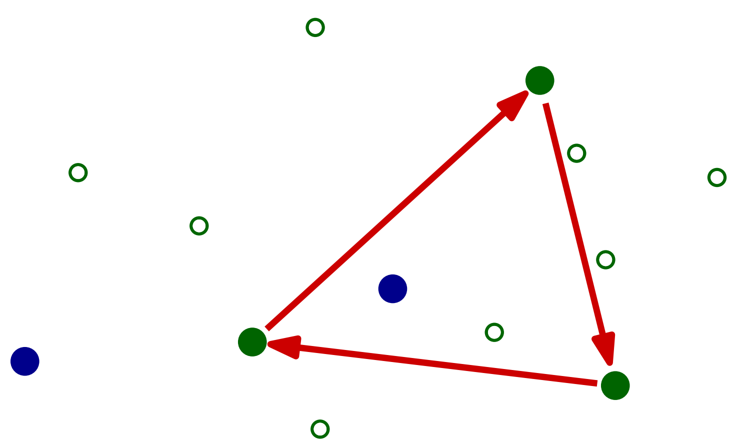

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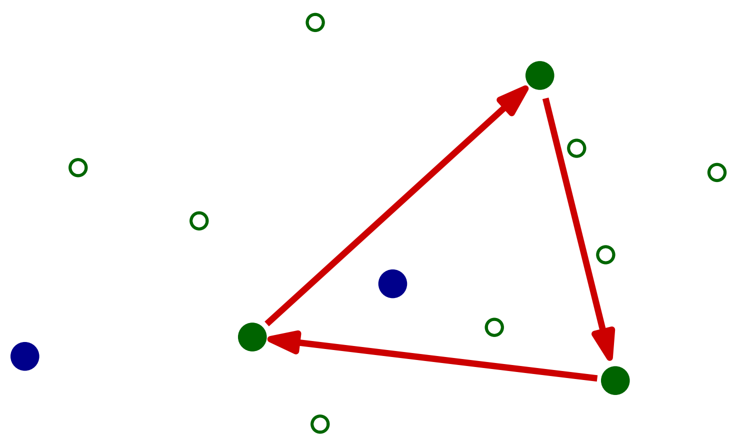

- in conflict with two configs

Randomized Incremental Construction: The Algorithm

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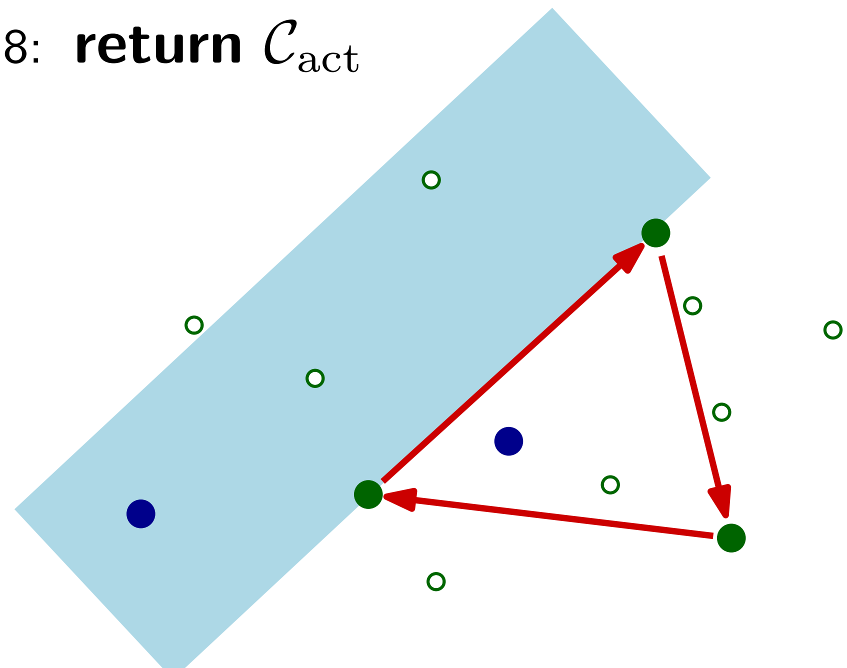

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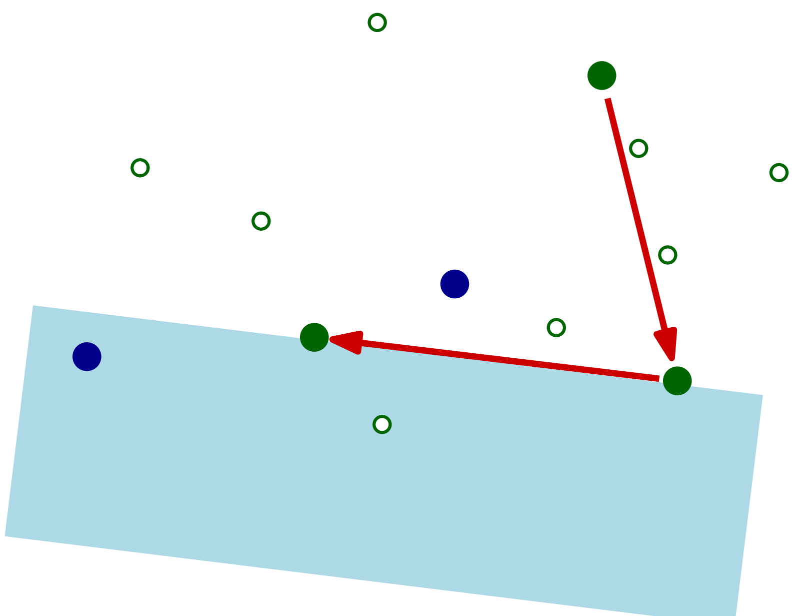

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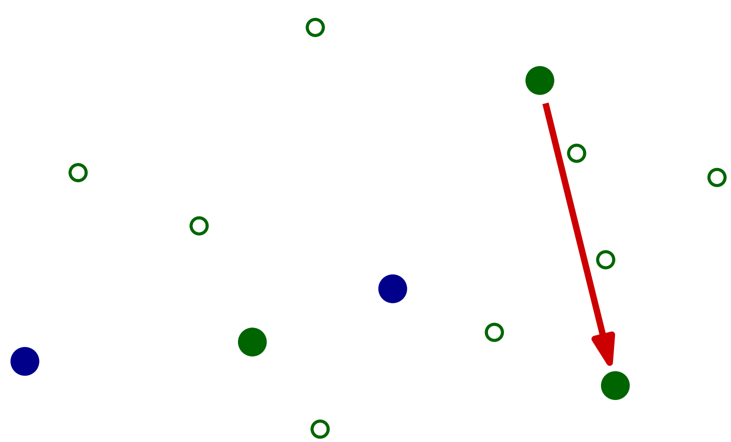

- in conflict with two configs
- two new configs appear

Randomized Incremental Construction: The Algorithm

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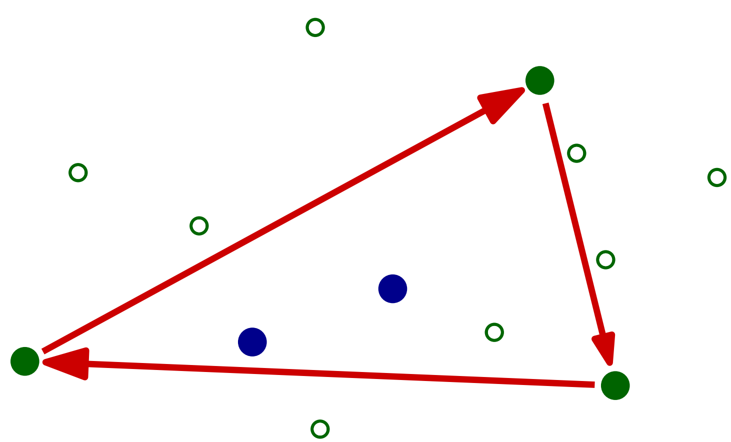

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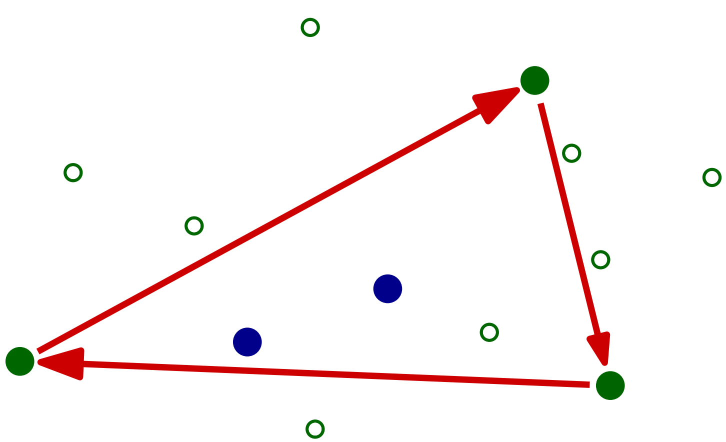

- in conflict with two configs
- two new configs appear

  conflict lists are subset of union of old conflict lists

Theorem. Let $S_j := \{x_1, \ldots, x_j\}$. Then

(i) $\mathrm{E}\left[\, |\mathcal{C}_{\text{act}}(S_j) \setminus \mathcal{C}_{\text{act}}(S_{j-1)}| \,\right] = O\left(\frac{\mathrm{E}[\text{size of } \mathcal{C}_{\text{act}}(S_j)]}{j}\right)$

(ii) The total size of the conflict lists of the active configurations appearing over the course of the algorithm is $O\left(\sum_{j=1}^{n} \frac{n}{j^2} \cdot \mathrm{E}\left[\, |\mathcal{C}_{\text{act}}(S_j)| \,\right]\right)$

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- $|\mathcal{C}_{\mathrm{act}}(S_j)| \leqslant j$
- total running time is linear in total size of all (dis)appearing conflict lists

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(i) $\mathrm{E}\left[\, |\mathcal{C}_{\mathrm{act}}(S_j) \setminus \mathcal{C}_{\mathrm{act}}(S_{j-1)}|\, \right] = O\left(\frac{\mathrm{E}[\text{size of } \mathcal{C}_{\mathrm{act}}(S_j)]}{j} \right)$

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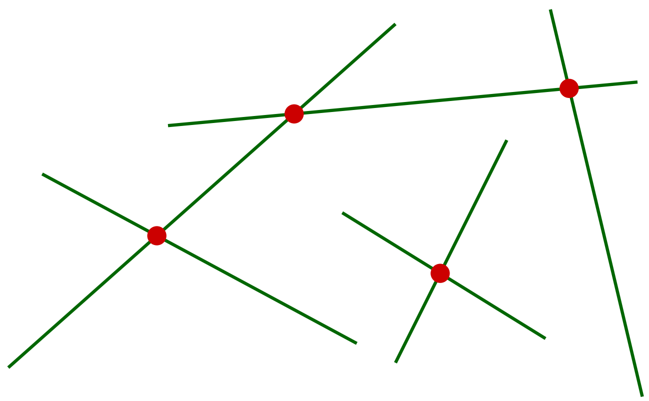
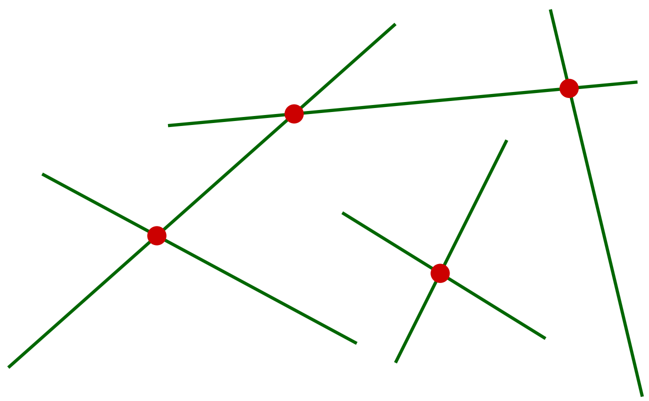
- $|\mathcal{C}_{\mathrm{act}}(S_j)| \leqslant j$
- total running time is linear in total size of all (dis)appearing conflict lists

convex-hull algorithm runs in $O(n \log n)$ time

Line-Segment Intersection with RIC





The framework

- S = set of n input objects
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    size should be bounded by a fixed constant
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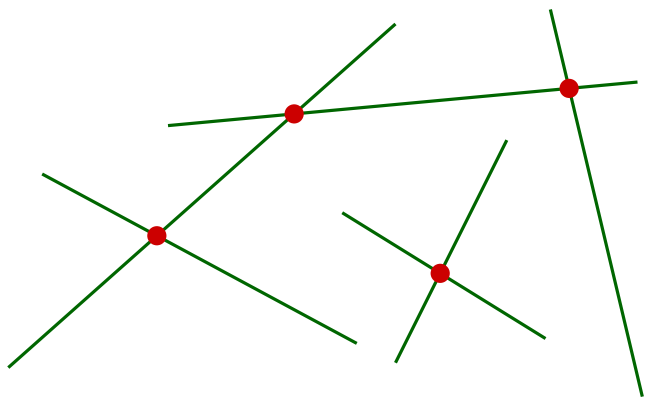
- Goal: Compute $\mathcal{C}_{\mathrm{act}}(S) = \{\Delta \in \mathcal{C}(S) : D(\Delta) \subseteq S \text{ and } K(\Delta) \cap S = \emptyset\}$



Configurations?

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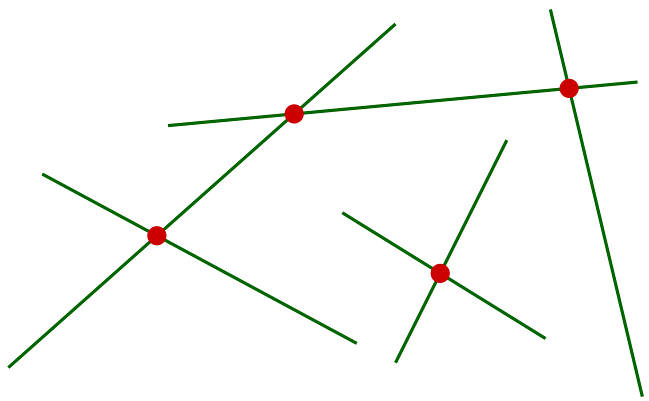

Configurations?

- intersection points does not work (find new configurations?)

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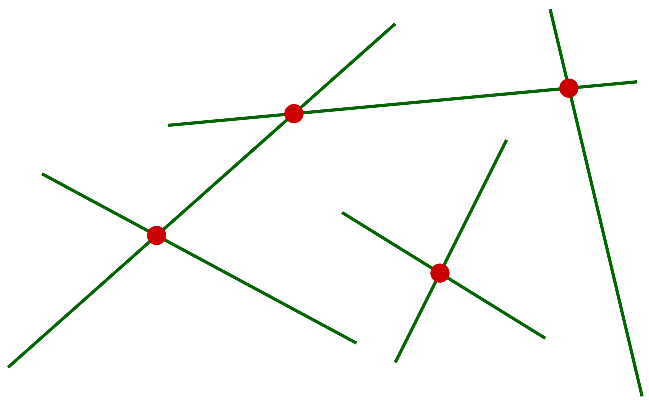

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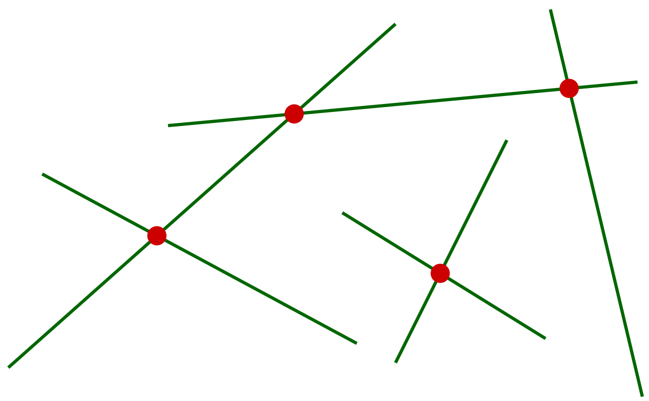

Configurations?

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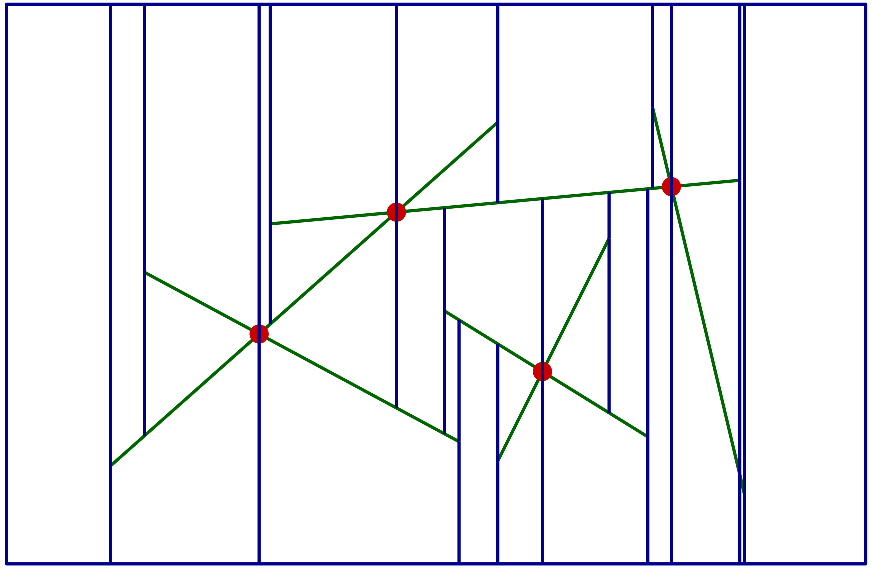

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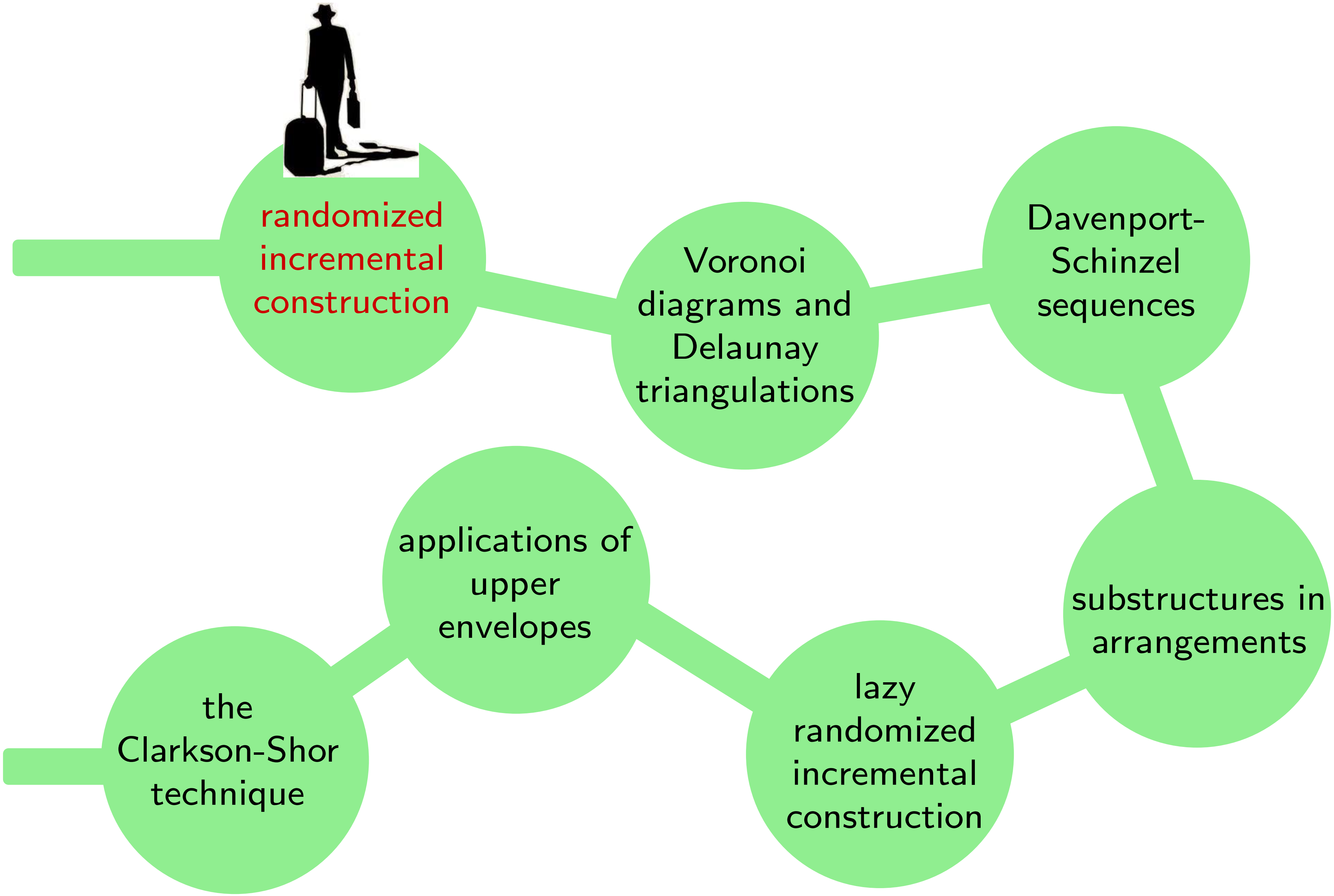
Course Overview



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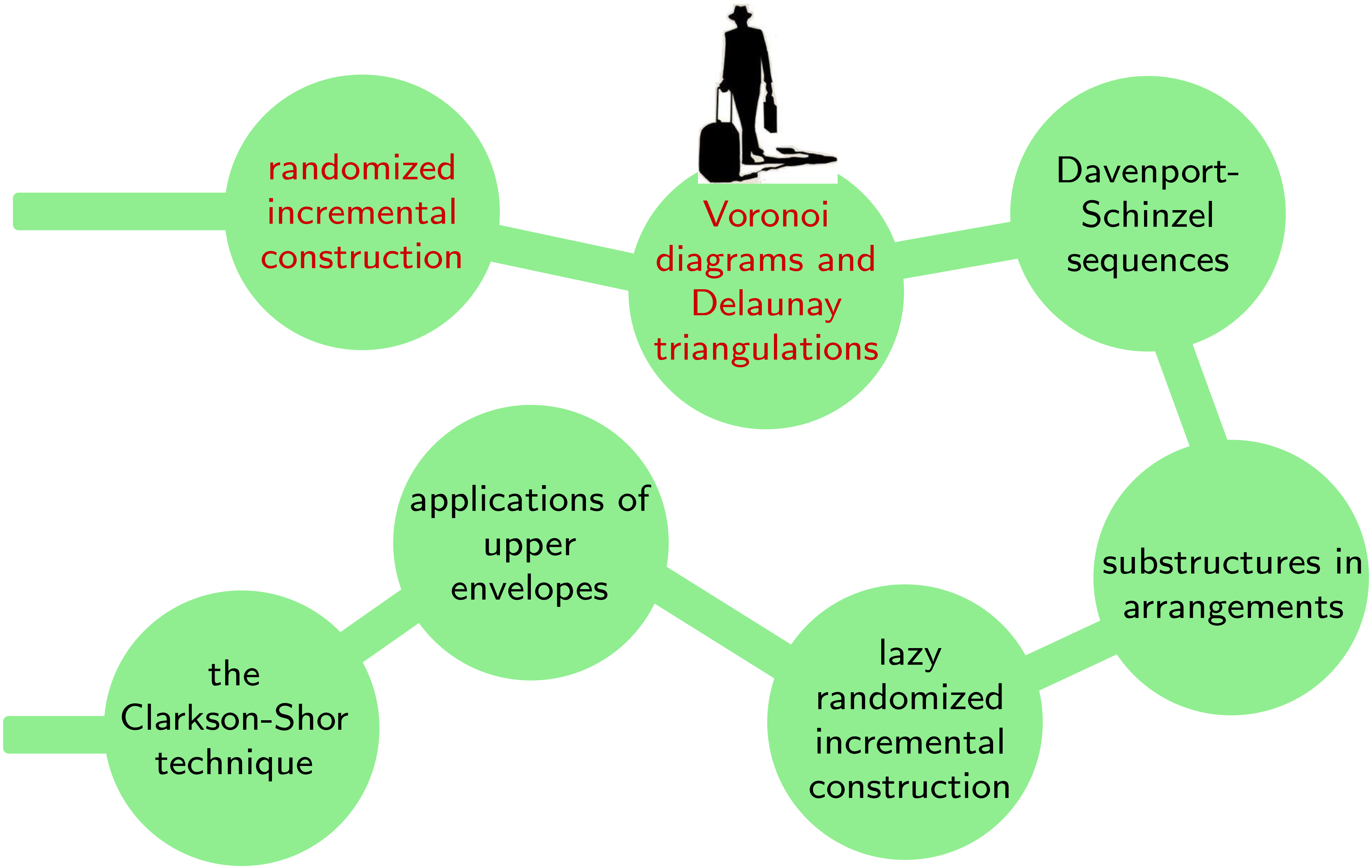


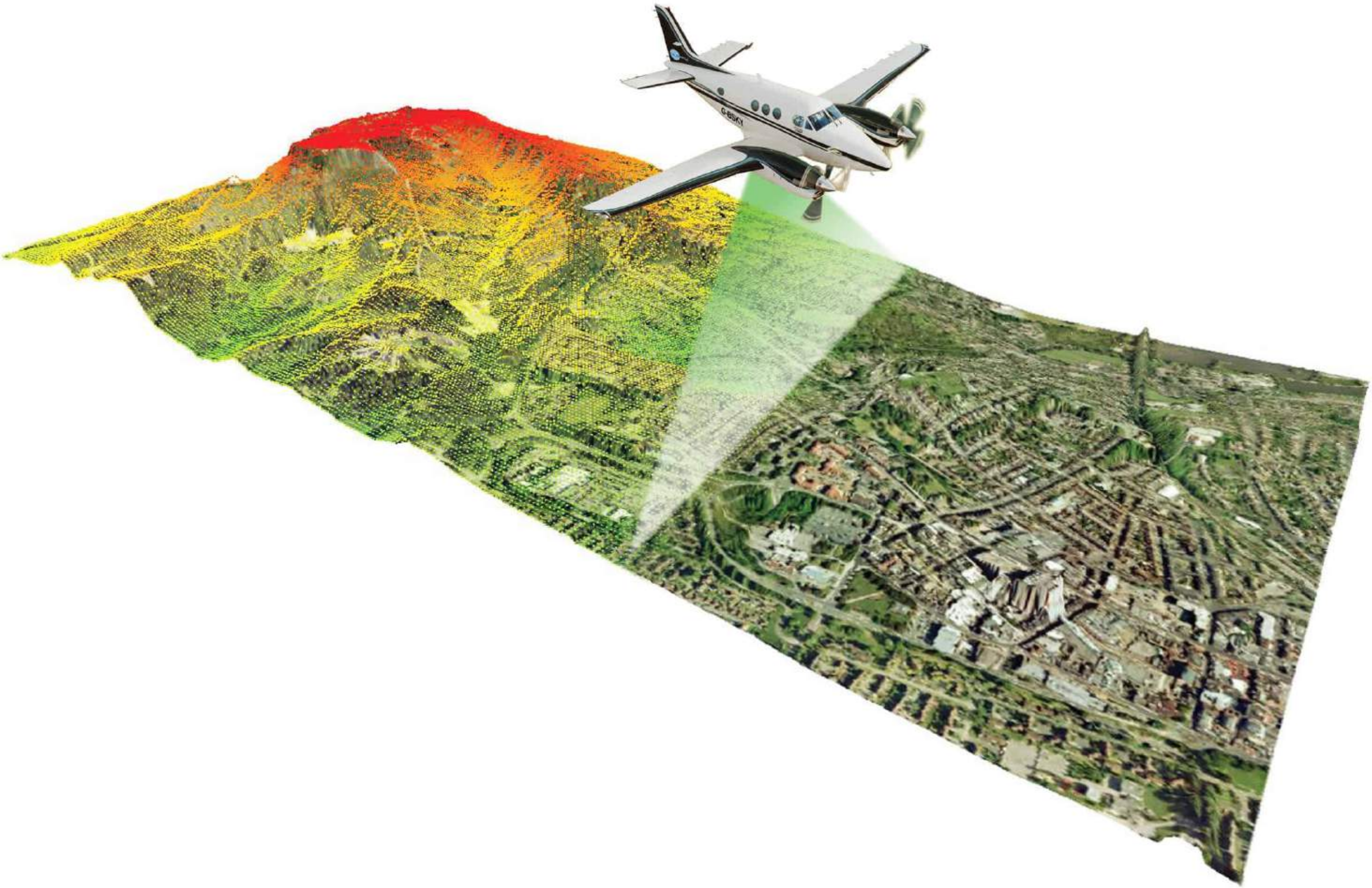
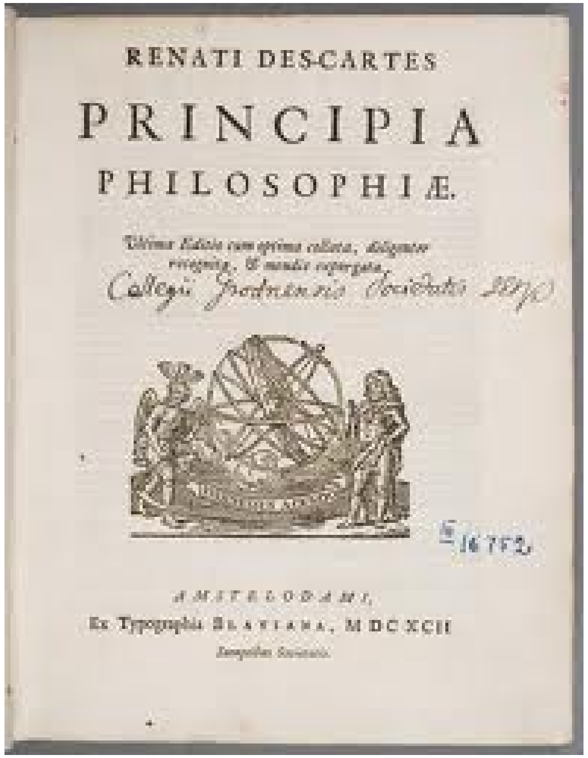


Image: www.aurorasolar.com

Principia Philosiphiae (Descartes, 1664)



RENATI DES-CARTES

PRINCIPIA

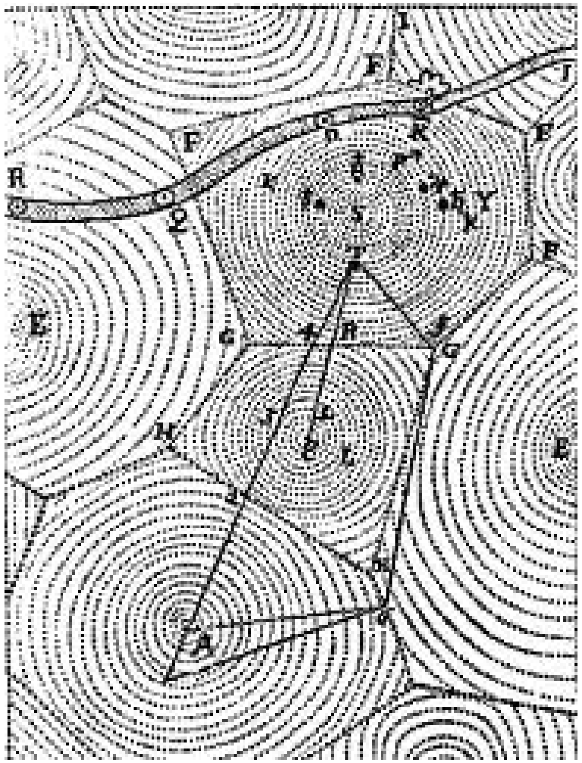
PHILOSOPHIÆ.

Ultima Editio cum optima collata, diligenter recognita, & mendis expurgata.

AMSTELODAMI,

Ex Typographia BLAVIANA, MDCXCII

Sumptibus Societatis.



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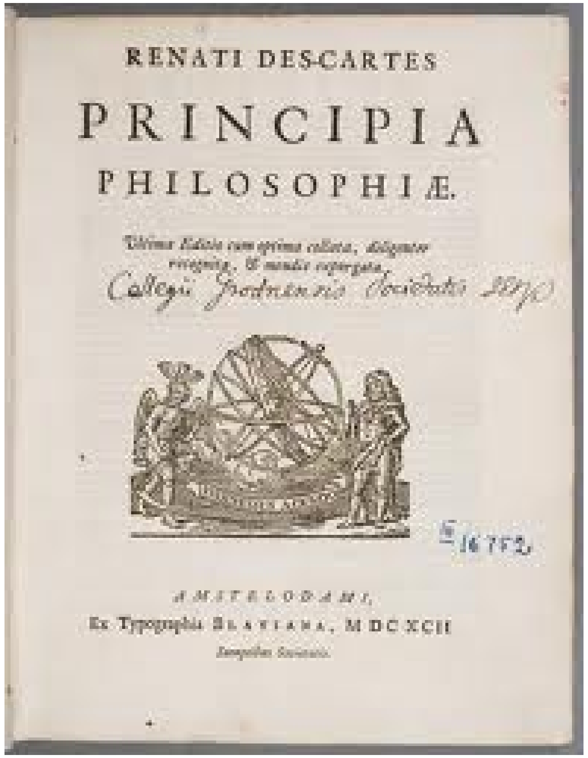

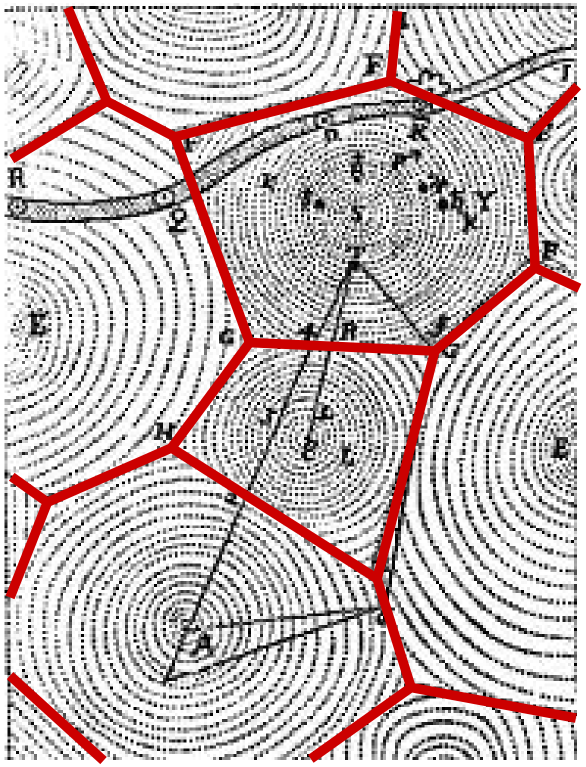

Voronoi diagram

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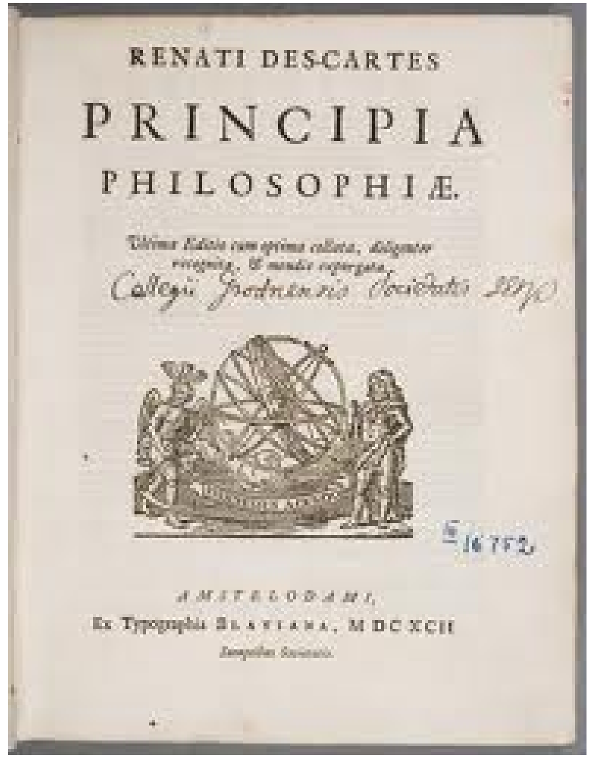

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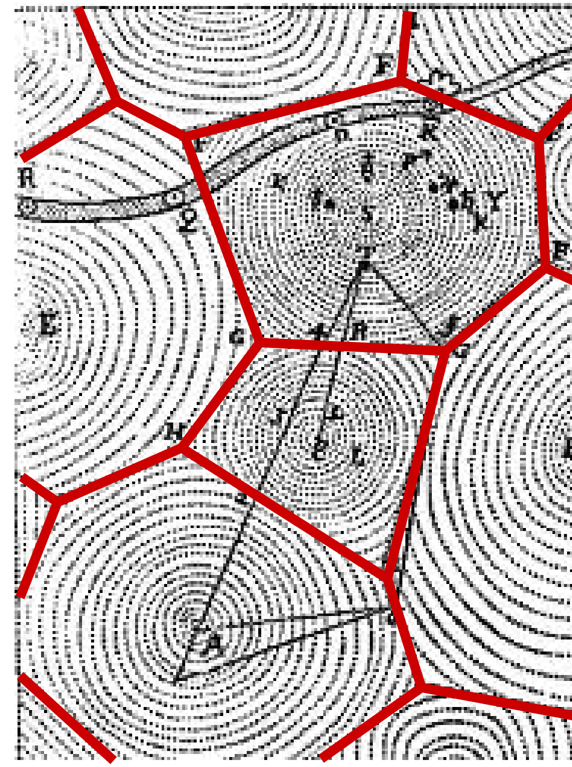
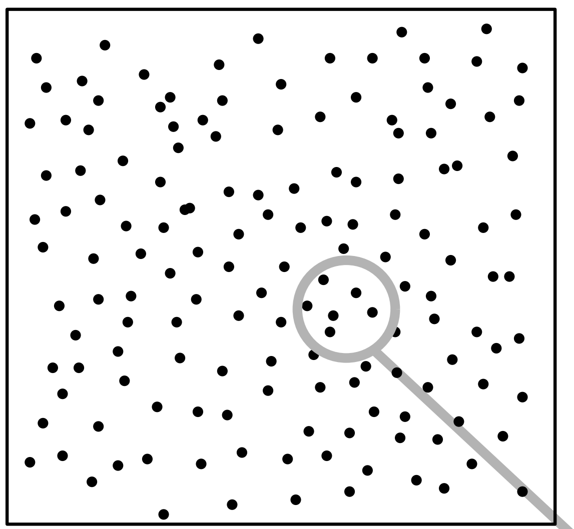


Voronoi diagram

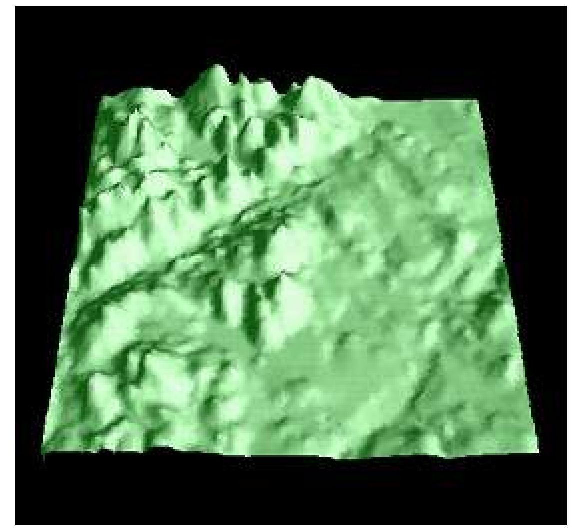
Georgy Voronoy
(1868-1908)

Terrain Reconstruction from Elevation Data

Back to terrain reconstruction . . .

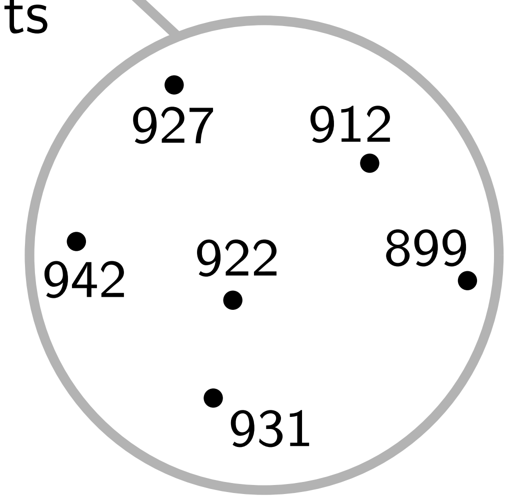


$\Longrightarrow$

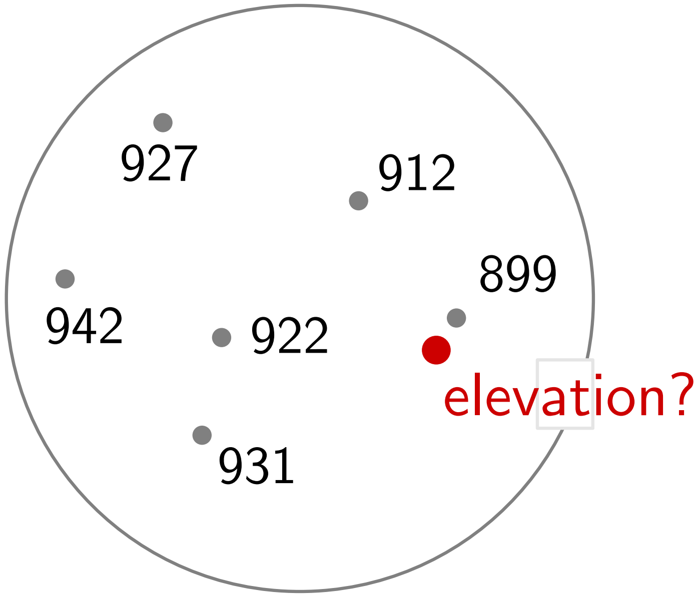


Input: terrain elevation at sample points

Goal: construct continuous terrain surface



Terrain Reconstruction from Elevation Data



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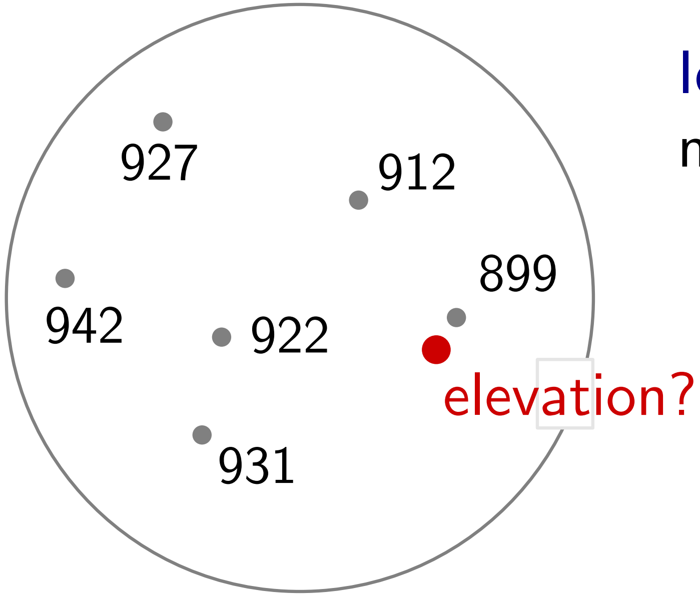
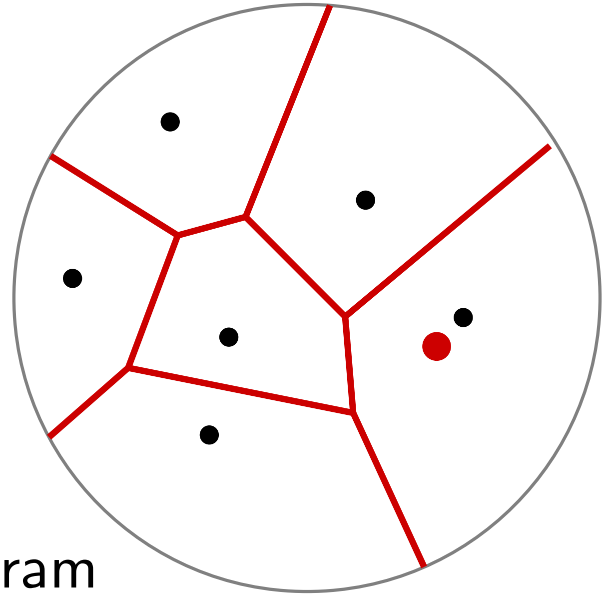
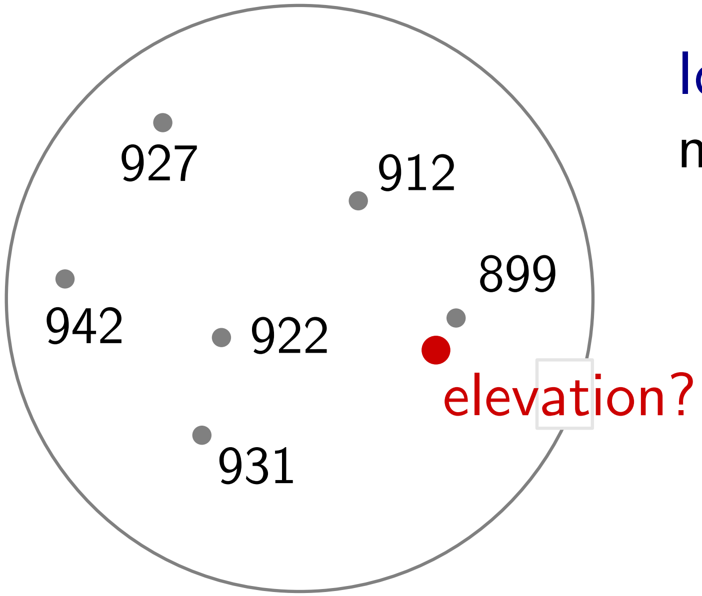


Idea: use elevation of nearest sample point



Voronoi diagram

Terrain Reconstruction from Elevation Data



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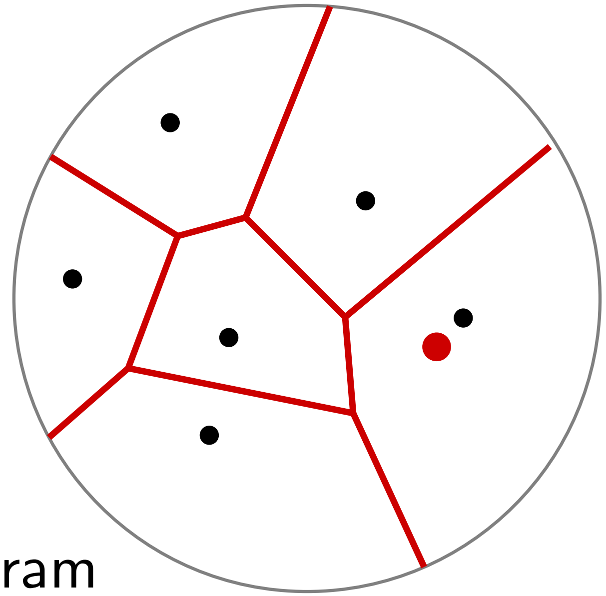
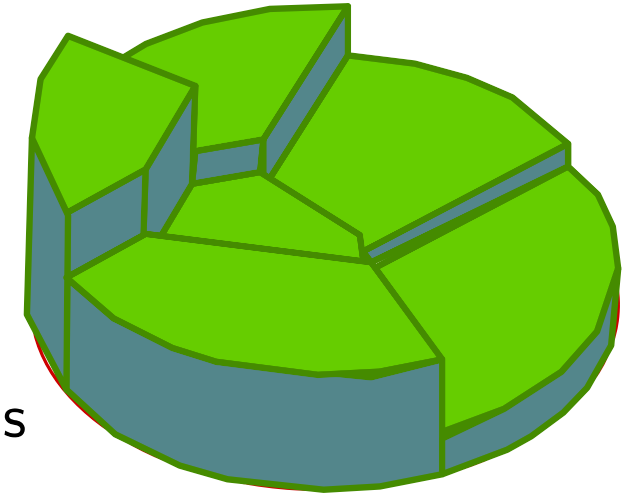


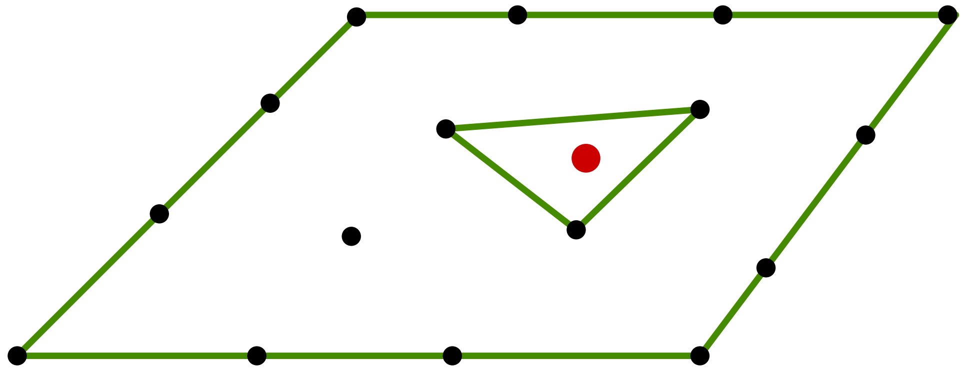
Voronoi diagram

Not good: surface not continuous



Terrain Reconstruction from Elevation Data

Better idea: determine elevation using interpolation



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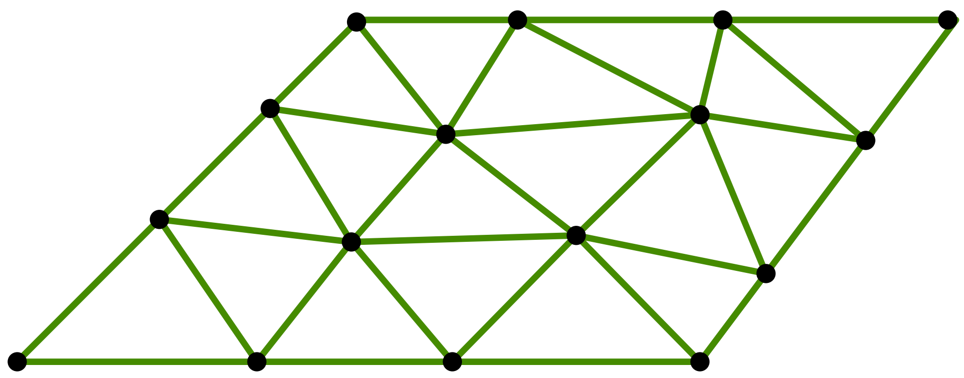

triangulation

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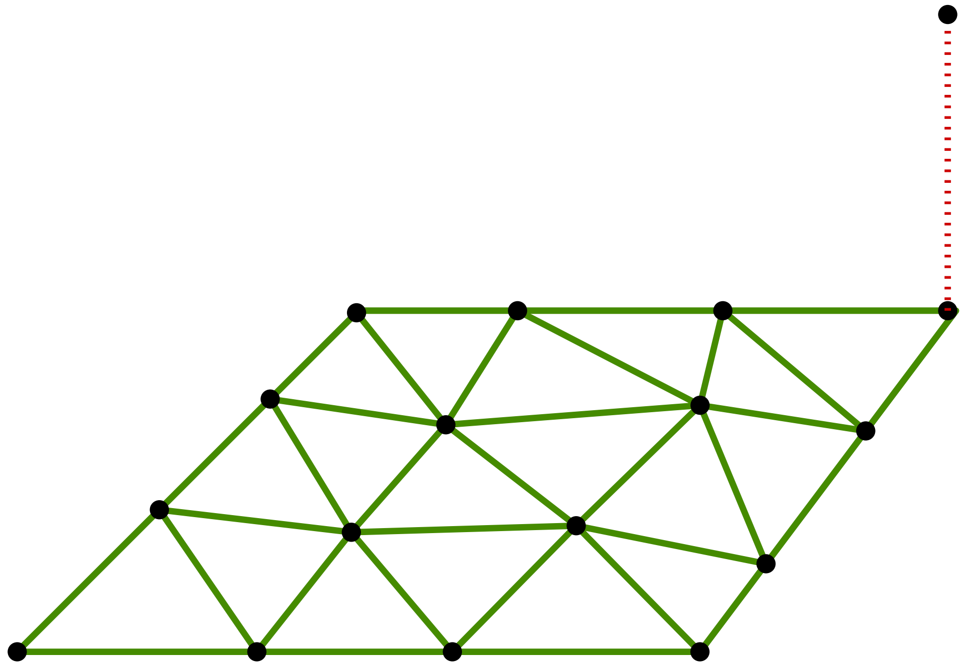

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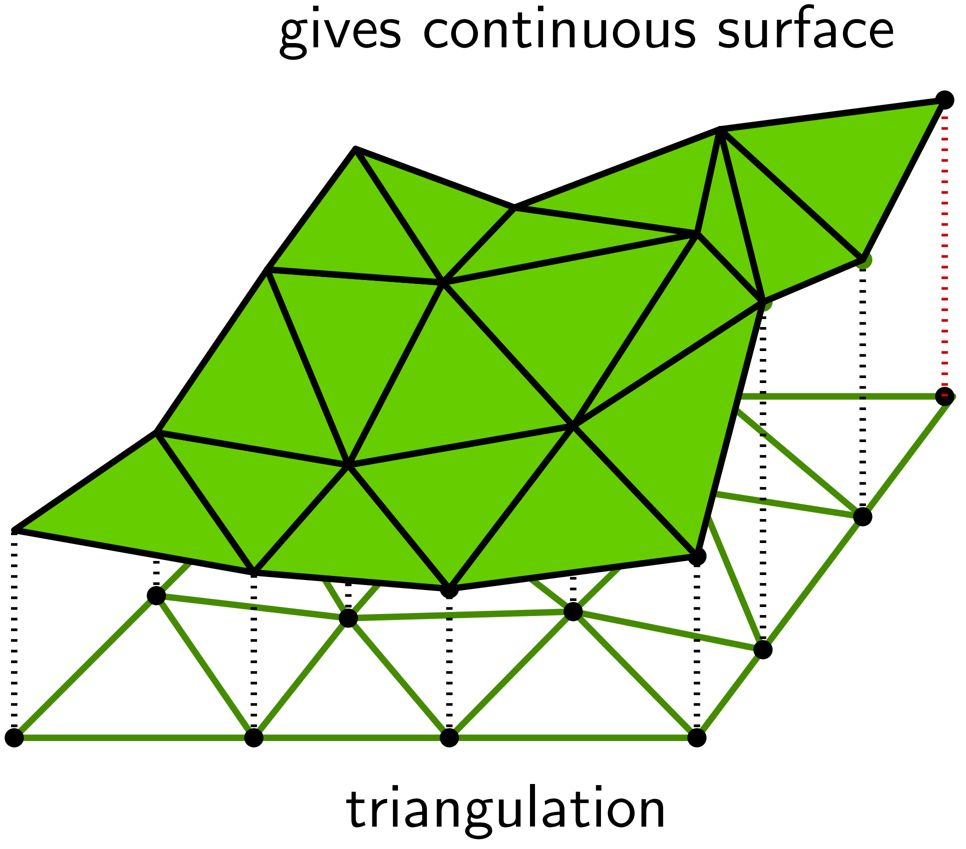
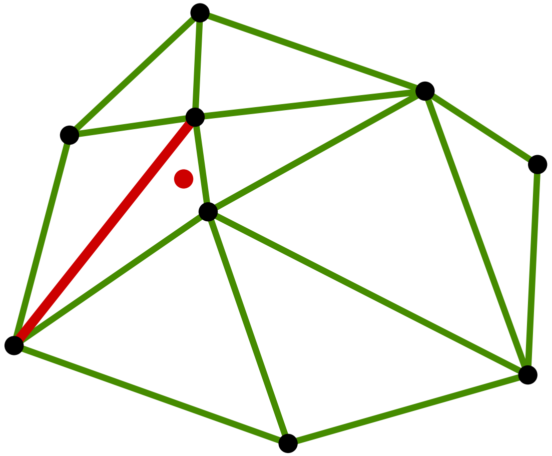
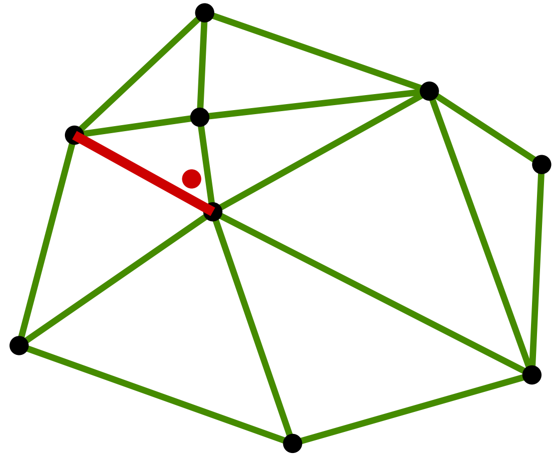


Which triangulation should we use?



or



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Terrain Reconstruction from Elevation Data

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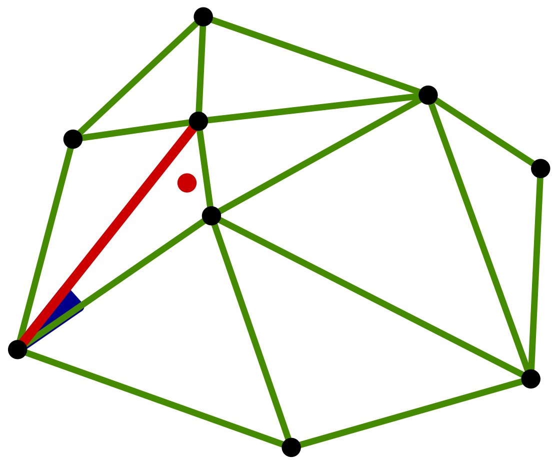

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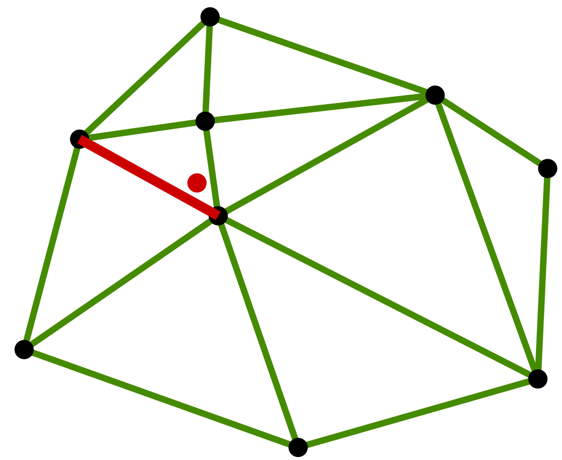


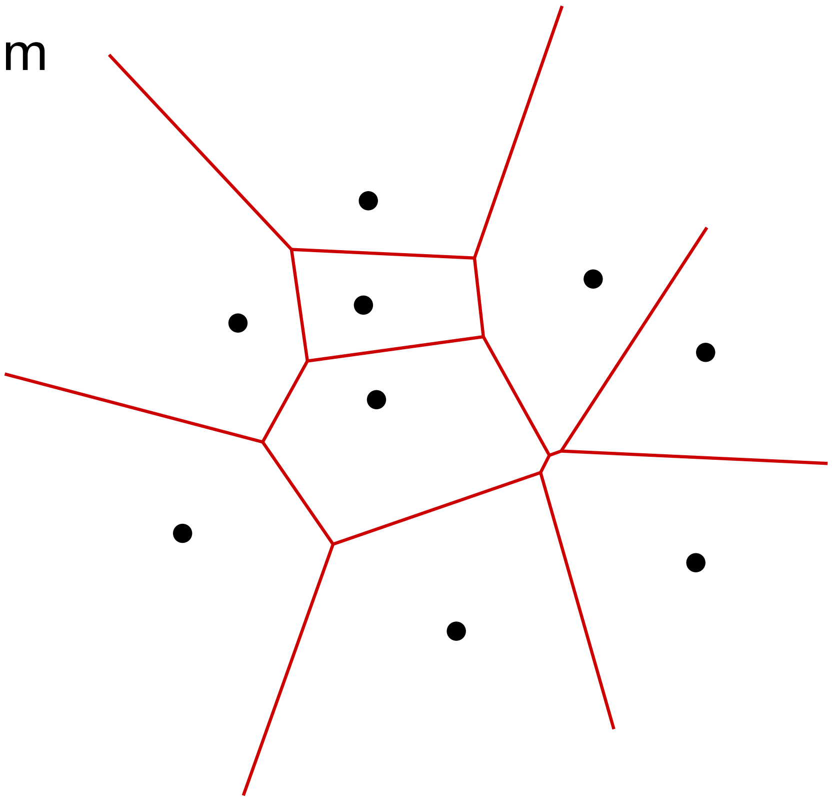
or . . .

long and thin triangles are bad $\Longrightarrow$ try to avoid small angles

Algorithmic problem: How can we quickly compute a triangulation that maximizes the minimum angle?

Terrain Reconstruction from Elevation Data

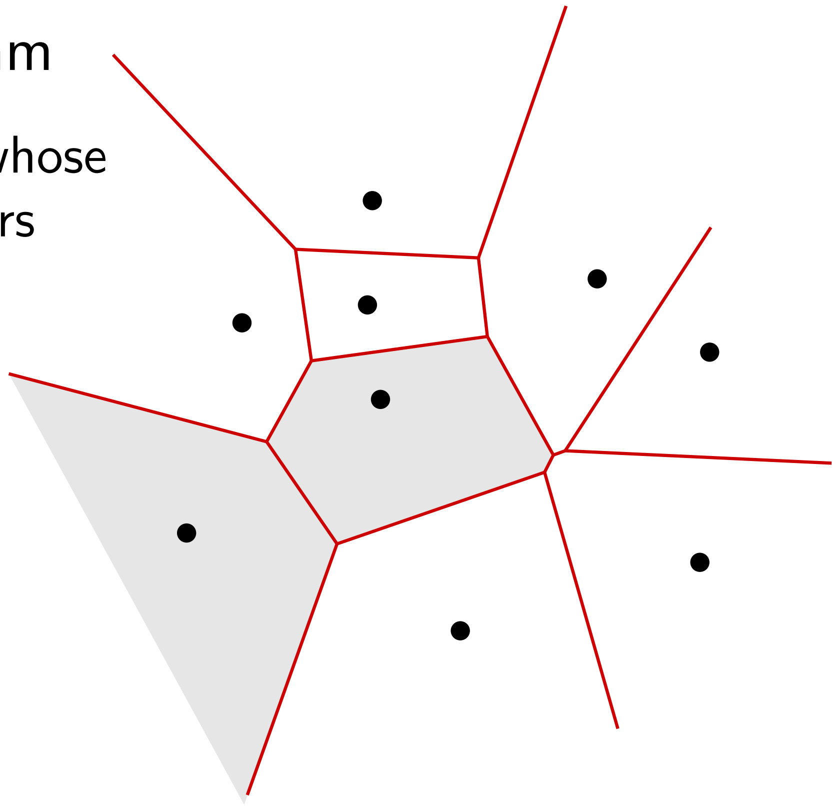
Voronoi diagram



Terrain Reconstruction from Elevation Data

Voronoi diagram

connect points whose
cells are neighbors



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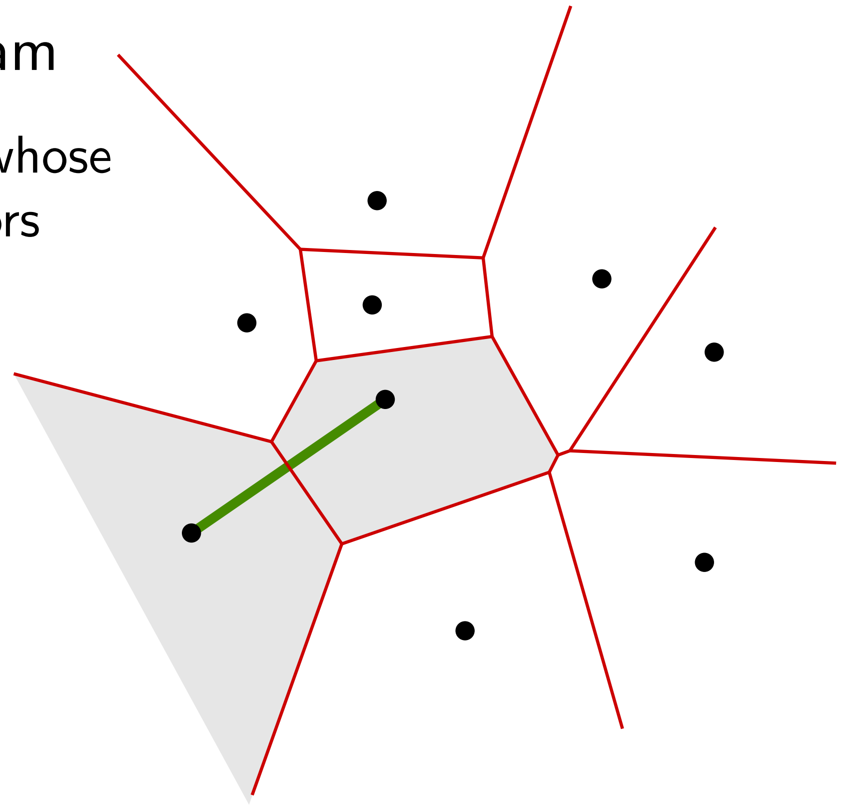

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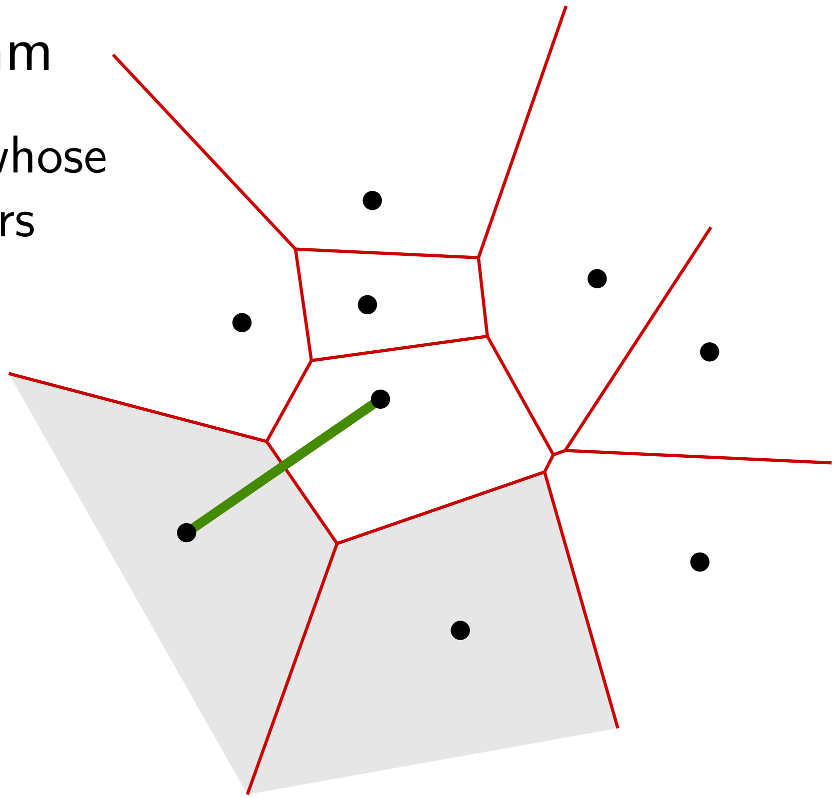

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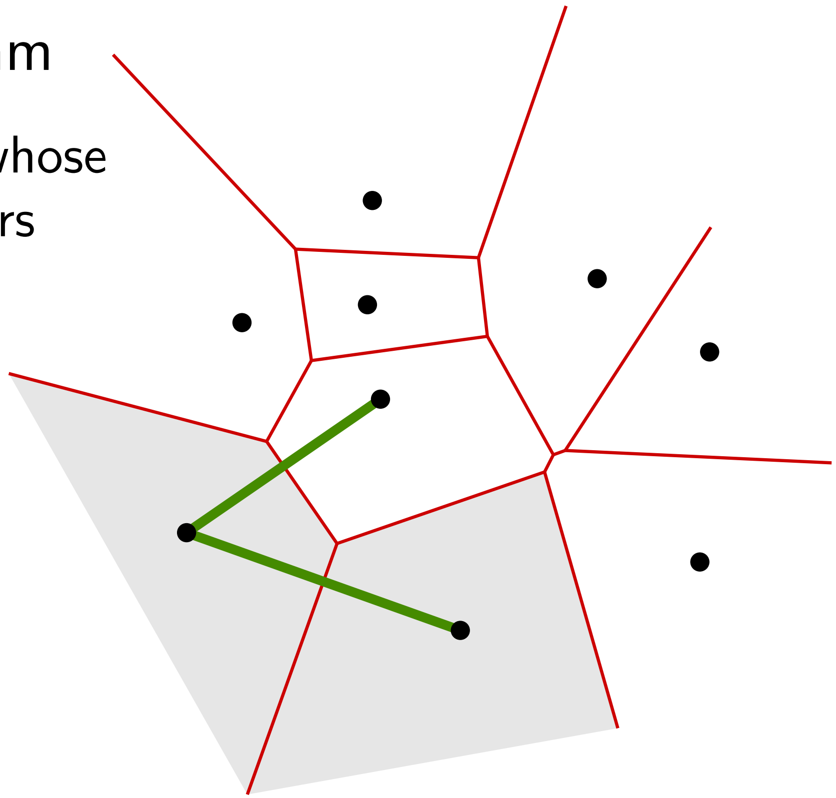

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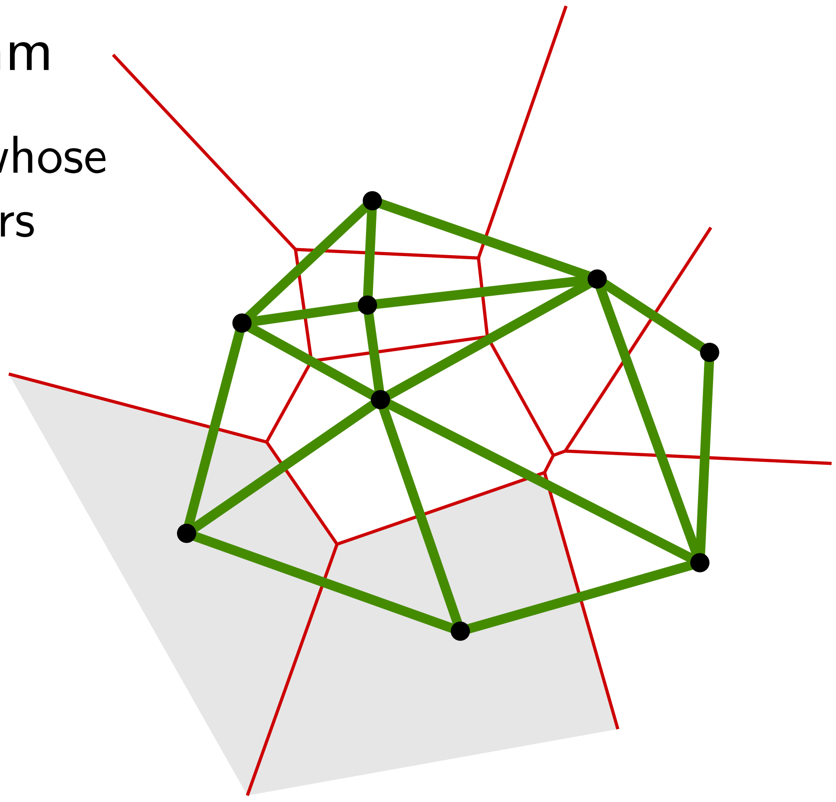

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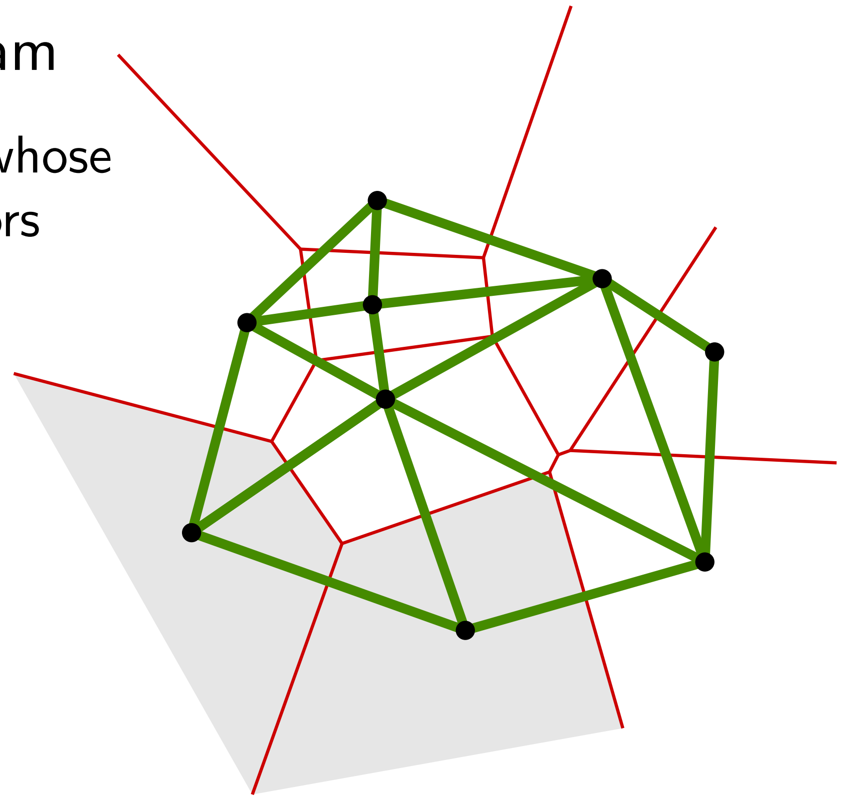


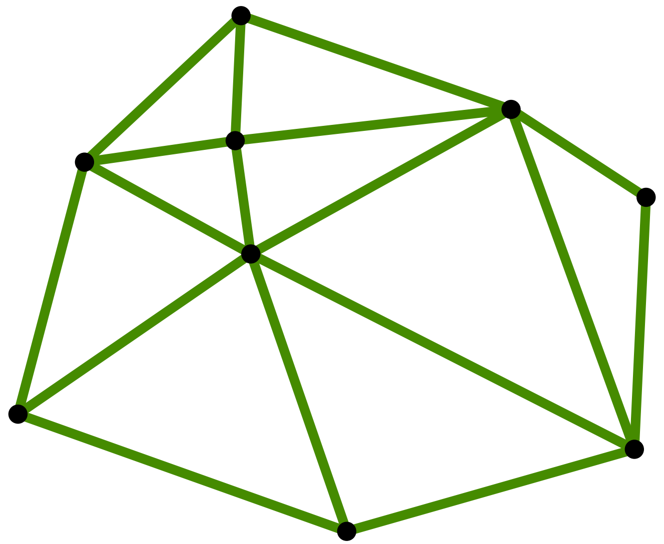
Delaunay triangulation:

triangulation that maximizes the minimum angle!

Boris Delaunay (1890 -1980)

Computing the Delaunay Triangulation

Computing the Delaunay Triangulation

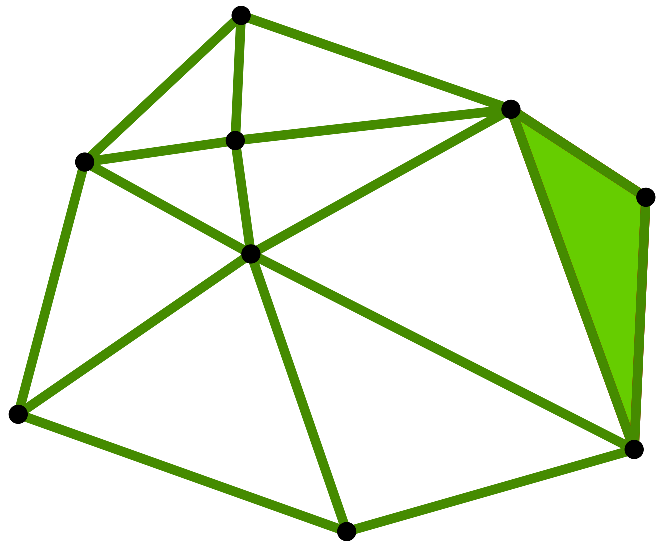


$\Delta(p, q, r)$ is in Delaunay triangulation

$$\Longleftrightarrow$$

Circle(p, q, r) contains no other point

Computing the Delaunay Triangulation



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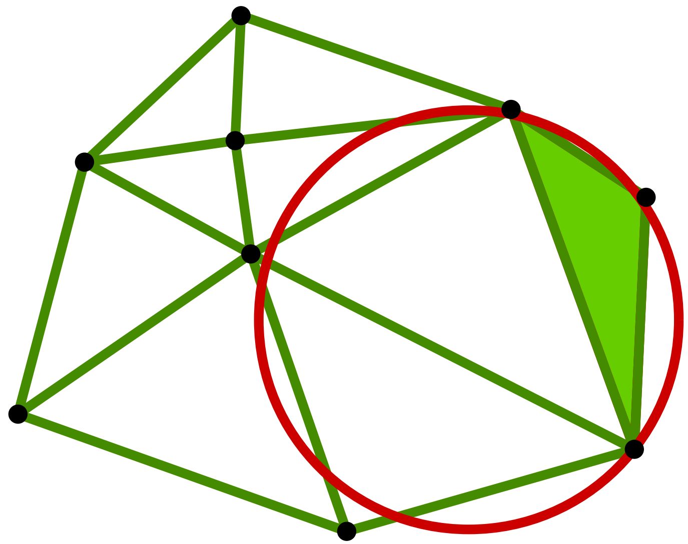

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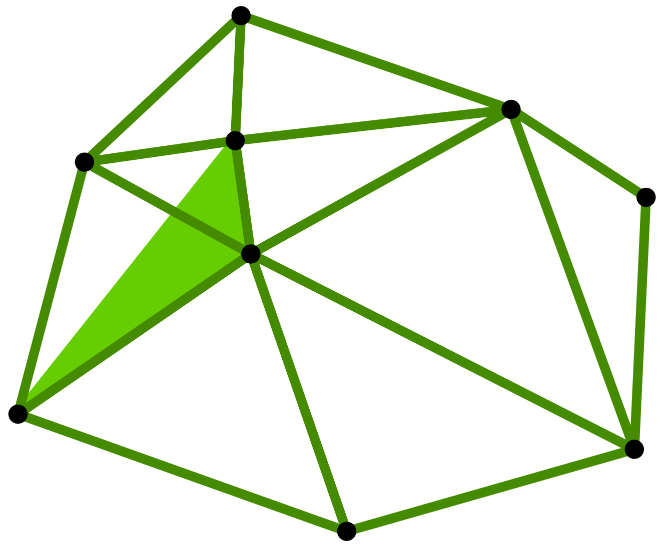

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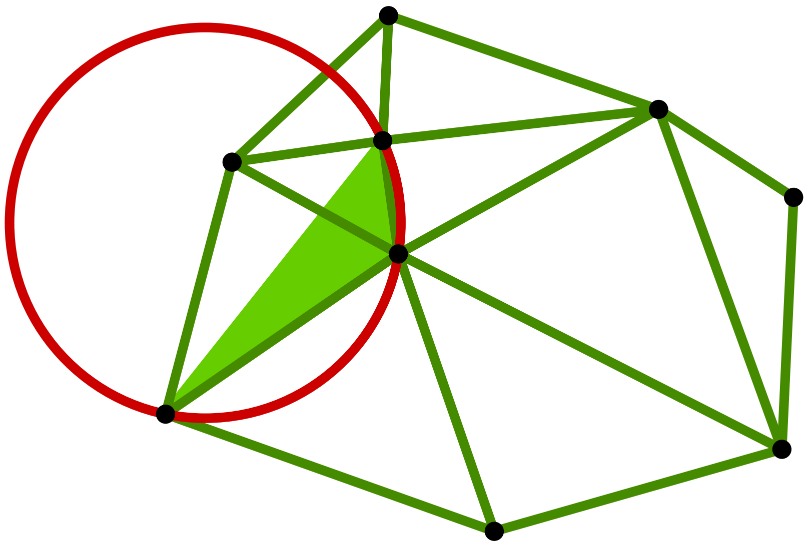

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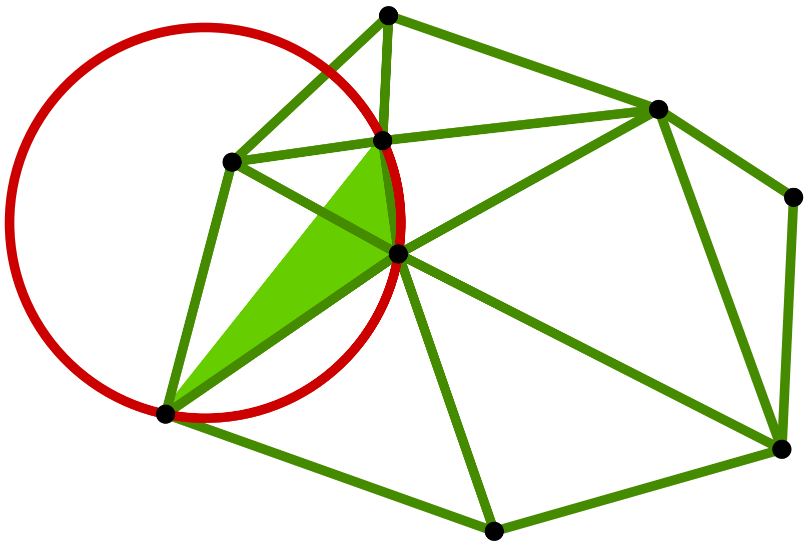

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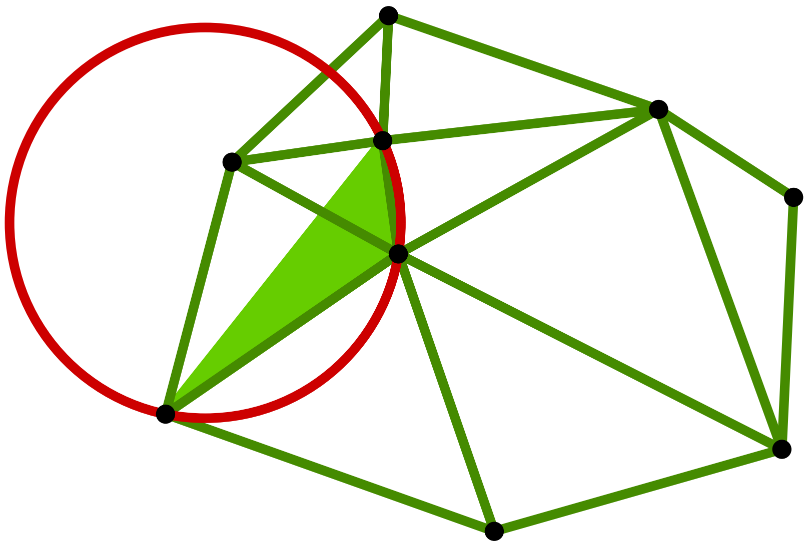
DELAUNAY-ALGORITHM(S)

1: $\mathcal{T} \leftarrow \emptyset$
2: **for** every triple of points p, q, r from S **do**
3:     **if** all other points from S lie outside Circle(p, q, r) **then**
4:         Add $\Delta(p, q, r)$ to $\mathcal{T}$
5: **return** $\mathcal{T}$

Computing the Delaunay Triangulation



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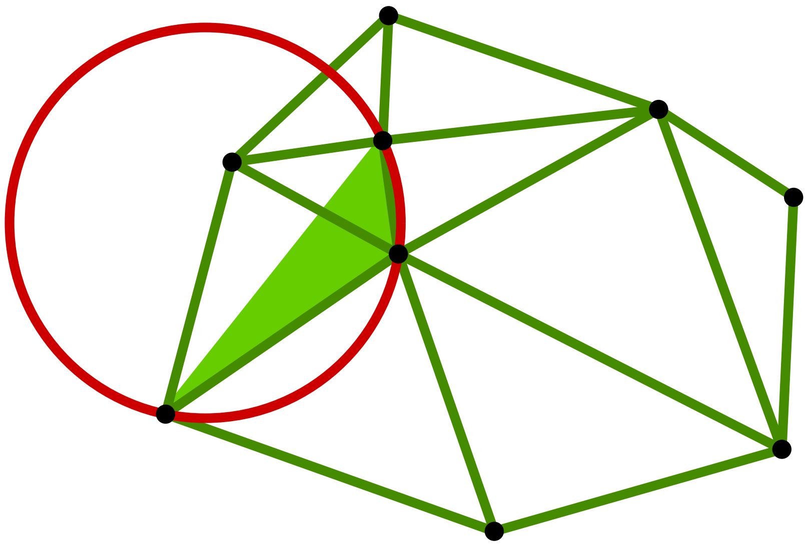
Running time:

Computing the Delaunay Triangulation



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Running time: $O(n^4)$

Computing the Delaunay Triangulation by RIC

Exercise

Apply the RIC framework to develop a randomized algorithm to compute the Delaunay triangulation, and analyze its running time.

Fact: The number of triangles in the Delaunay triangulation of a set S of n points in the plane is at most $2n - 5$.

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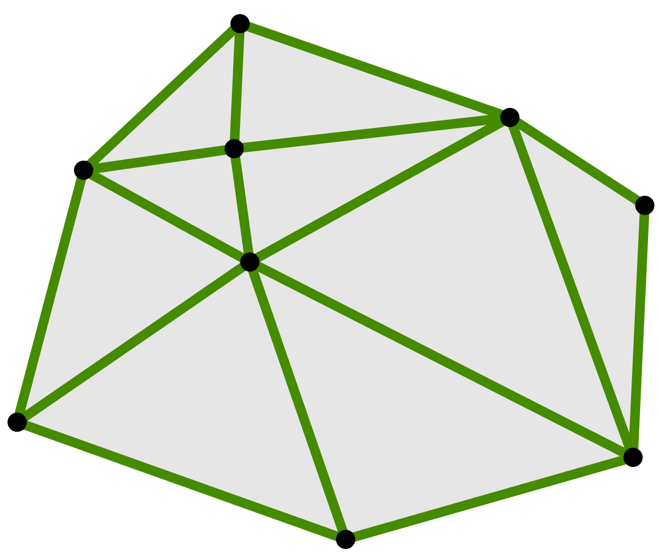

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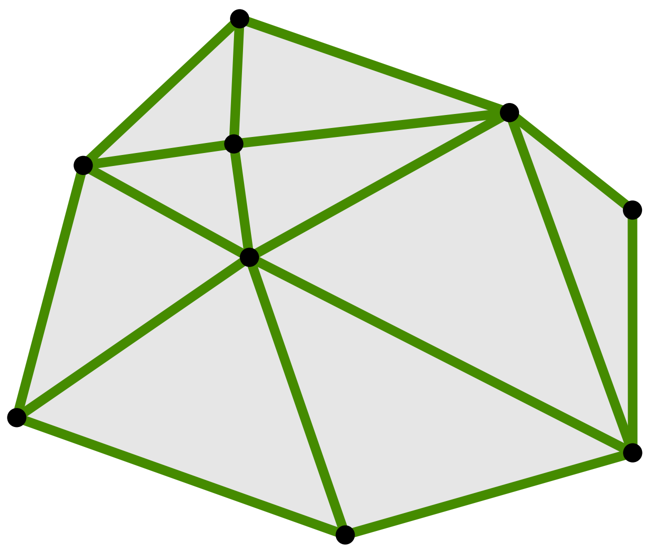
Computing the Delaunay Triangulation by RIC



- S = set of n input objects
- $\mathcal{C}(S)$ = set of configurations defined by S
  - $D(\Delta) \subset S$ = defining set of $\Delta \in \mathcal{C}(S)$
    size bounded by fixed constant
  - $K(\Delta) \subset S$ = conflict list of $\Delta \in \mathcal{C}(S)$
- Goal: Compute $\mathcal{C}_{\mathrm{act}}(S) = \{\Delta \in \mathcal{C}(S) : D(\Delta) \subseteq S$ and $K(\Delta) \cap S = \emptyset\}$

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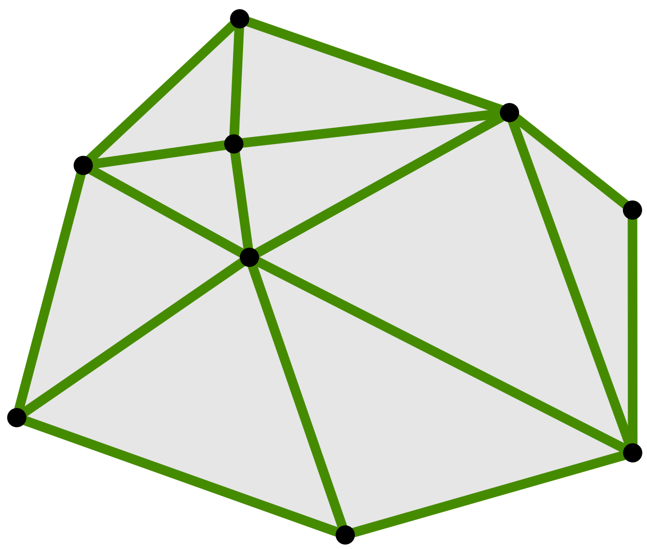

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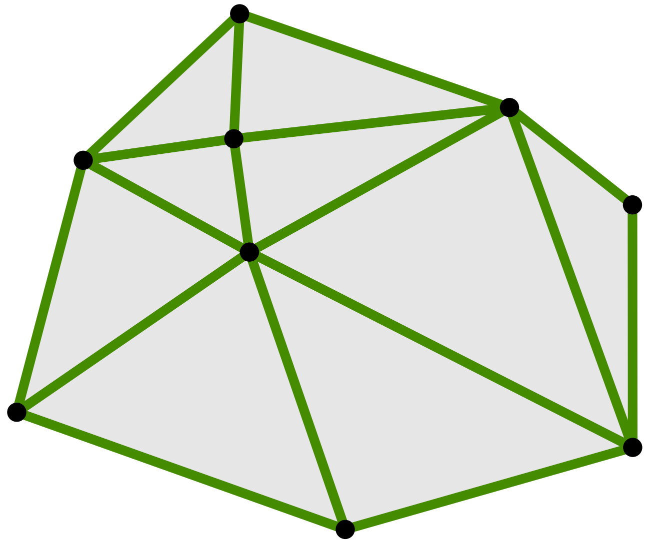

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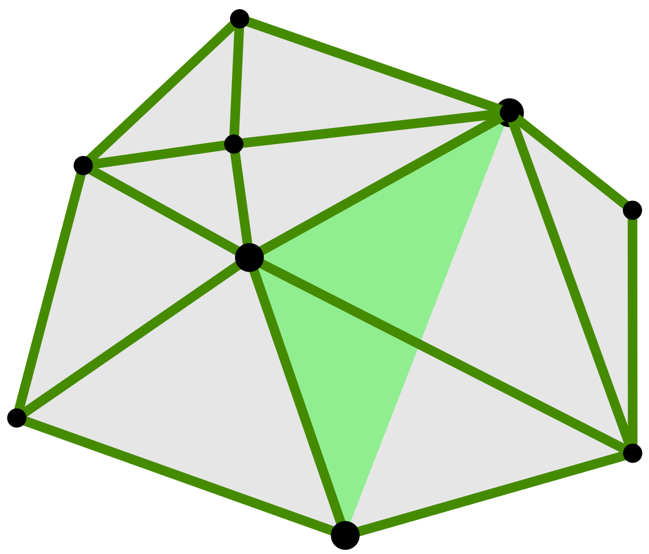

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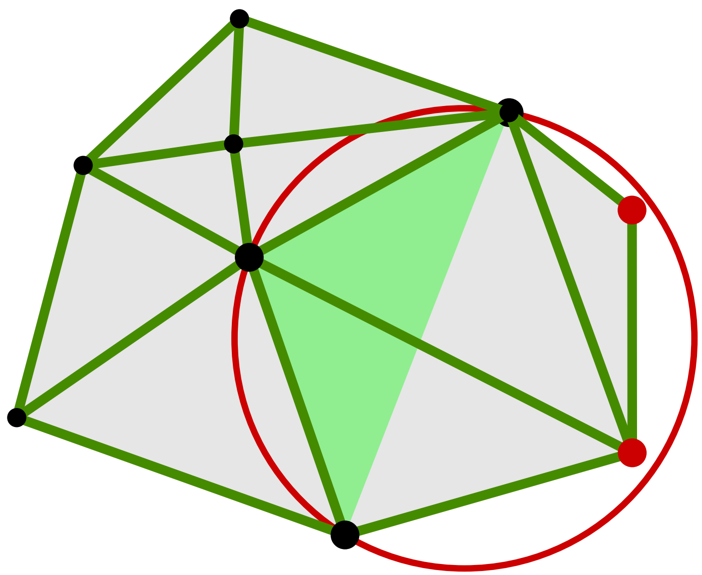

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Computing the Delaunay Triangulation by RIC

RIC-DELAUNAY(S)

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Intitialize conflict lists of configurations Δ ∈ C_act
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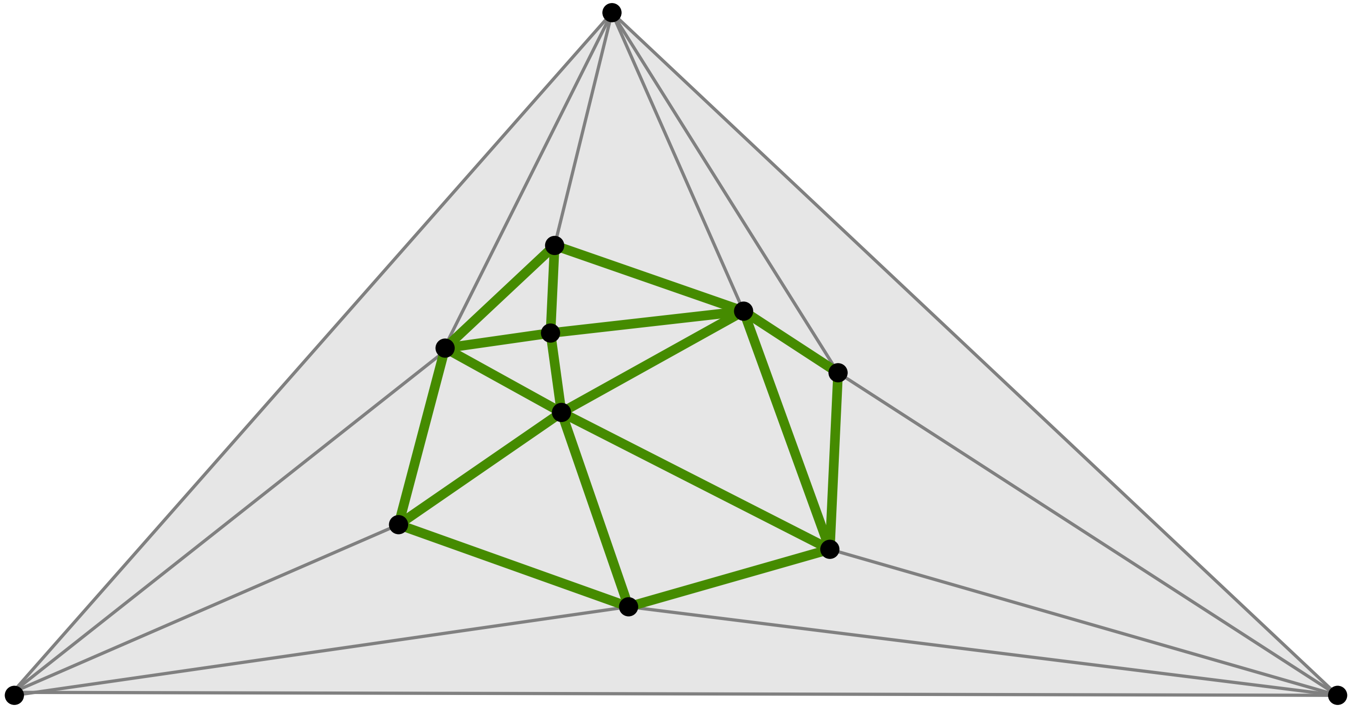

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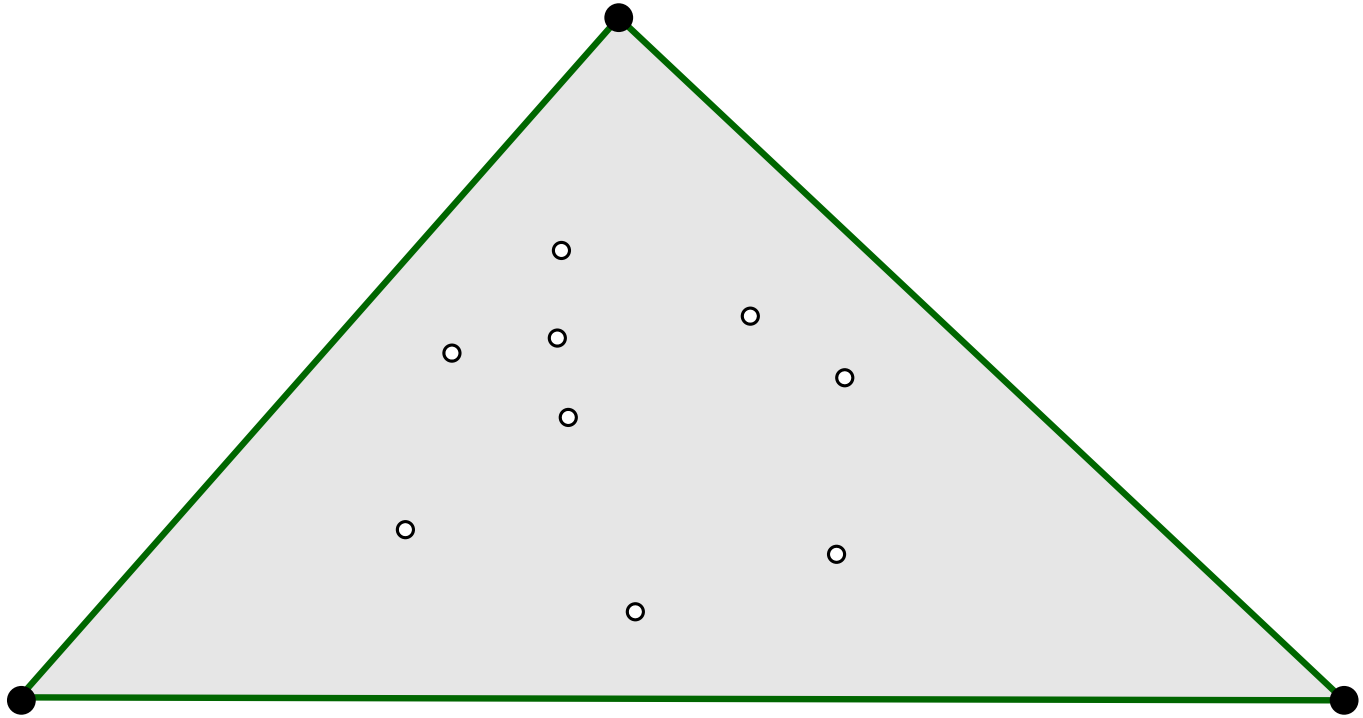

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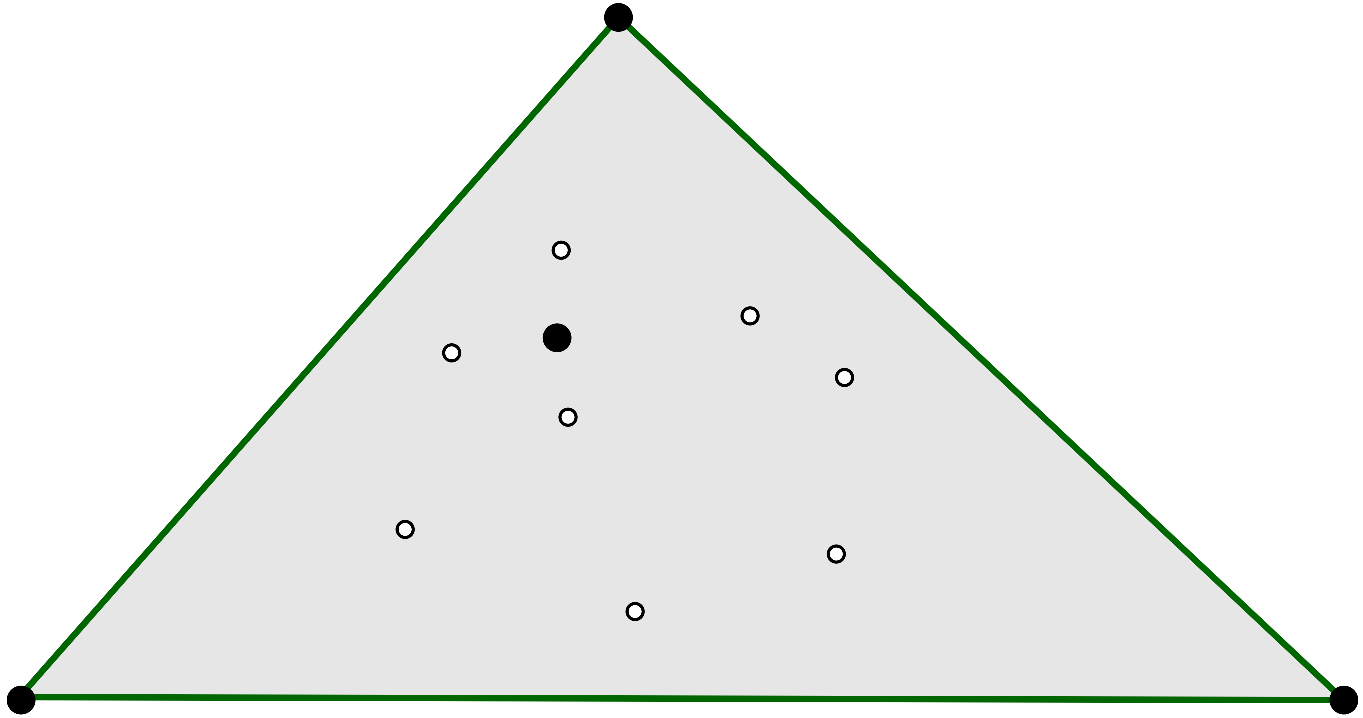

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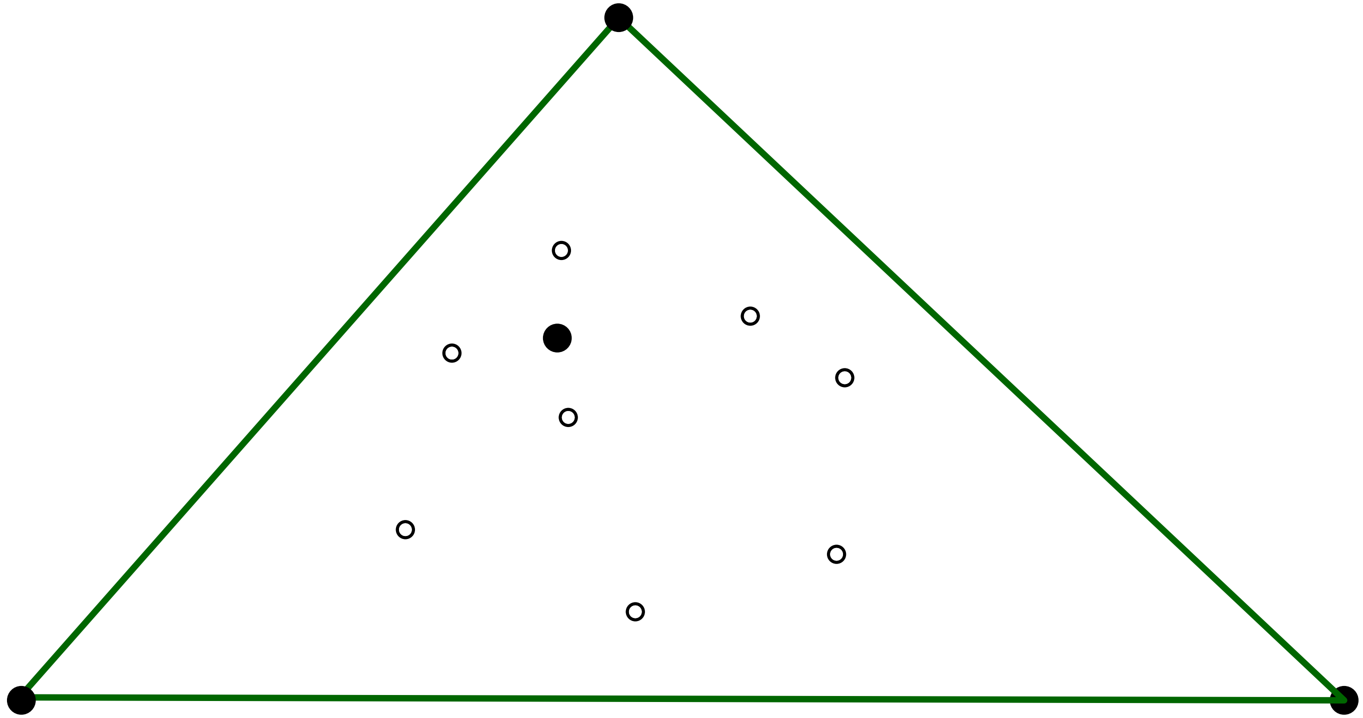

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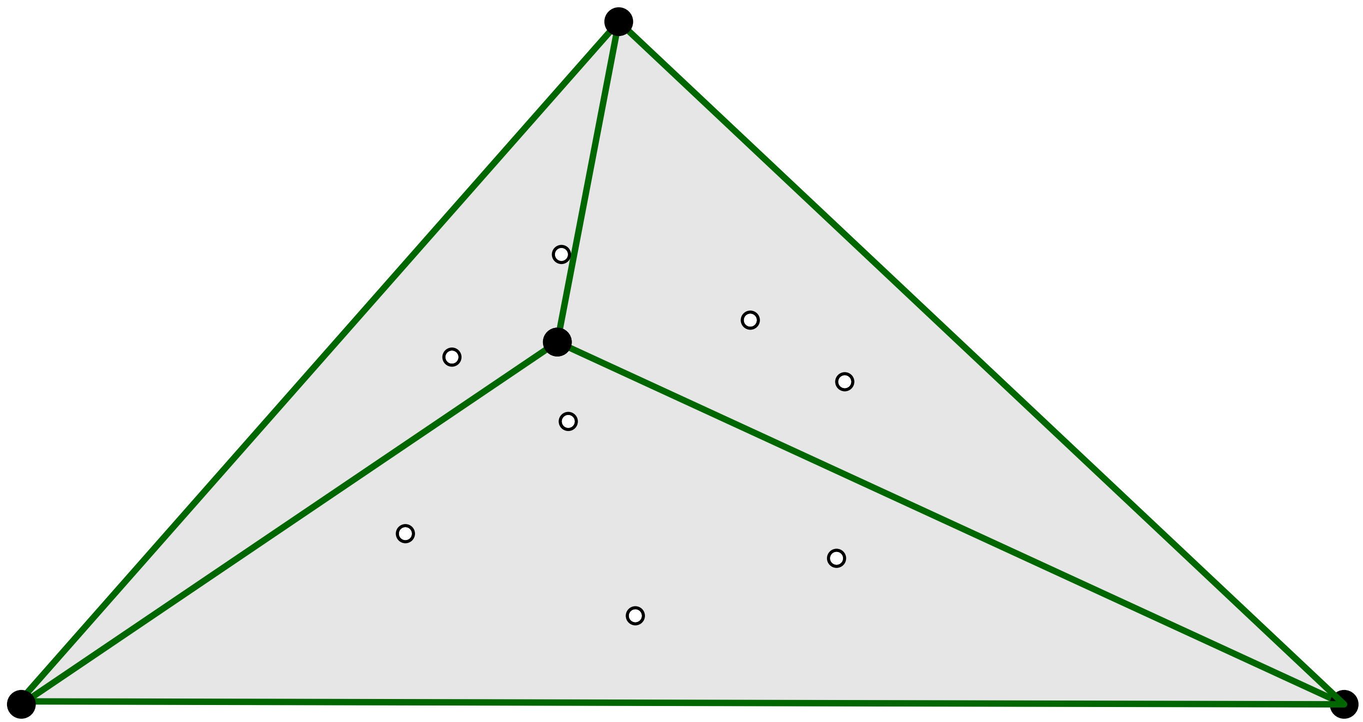

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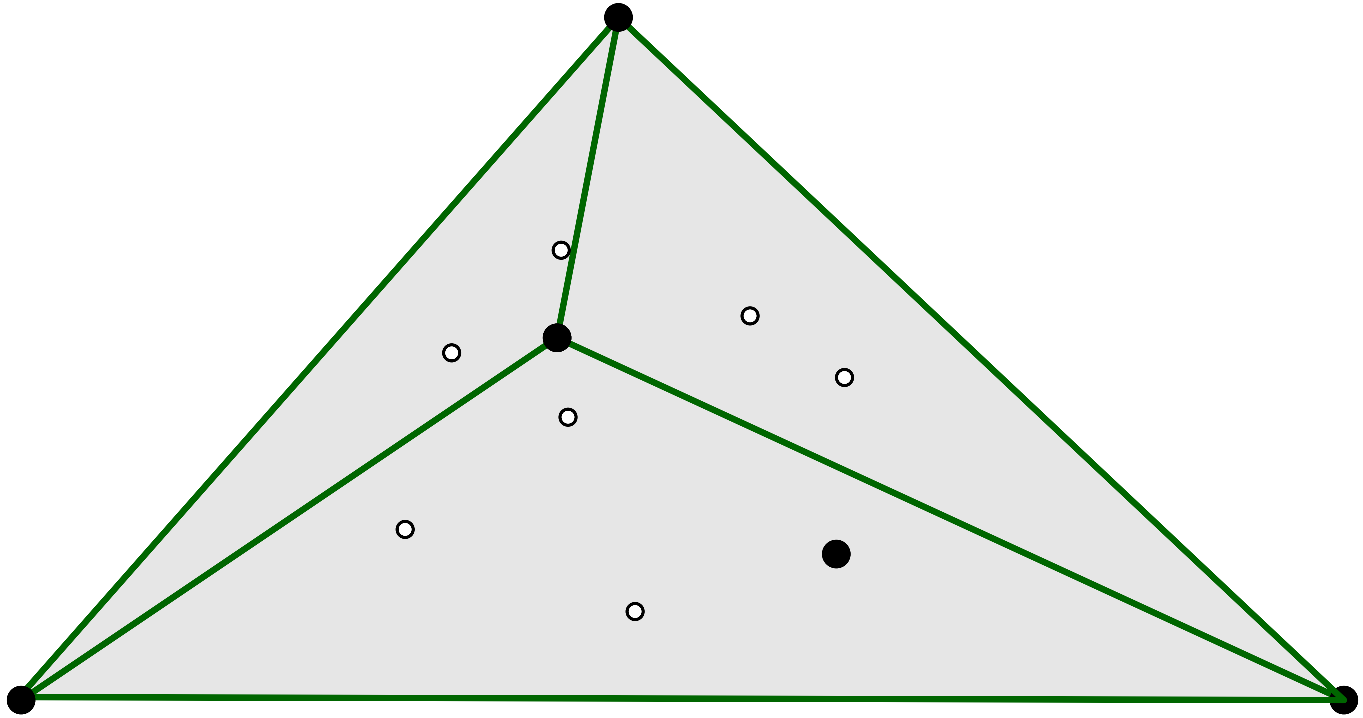

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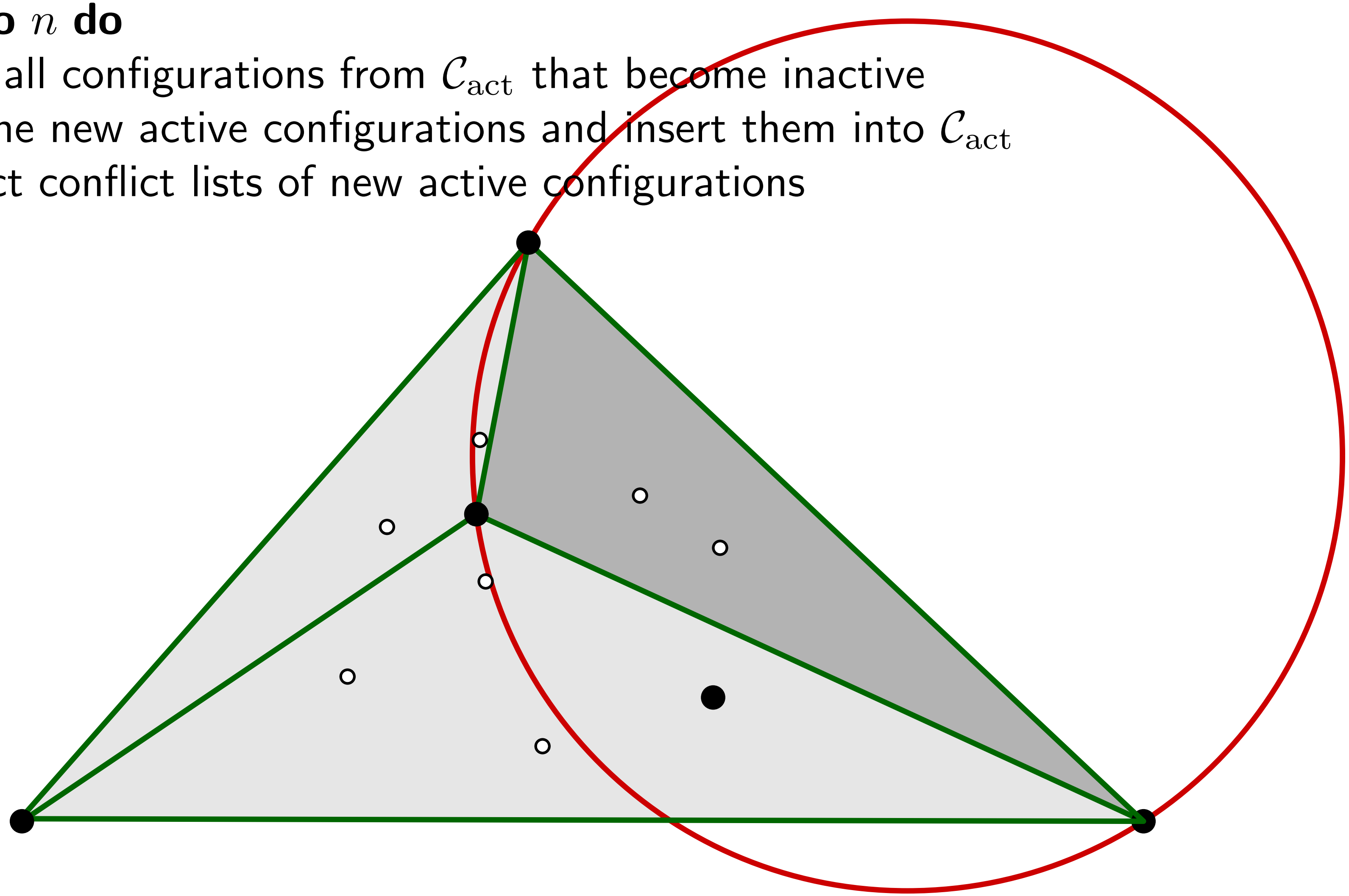

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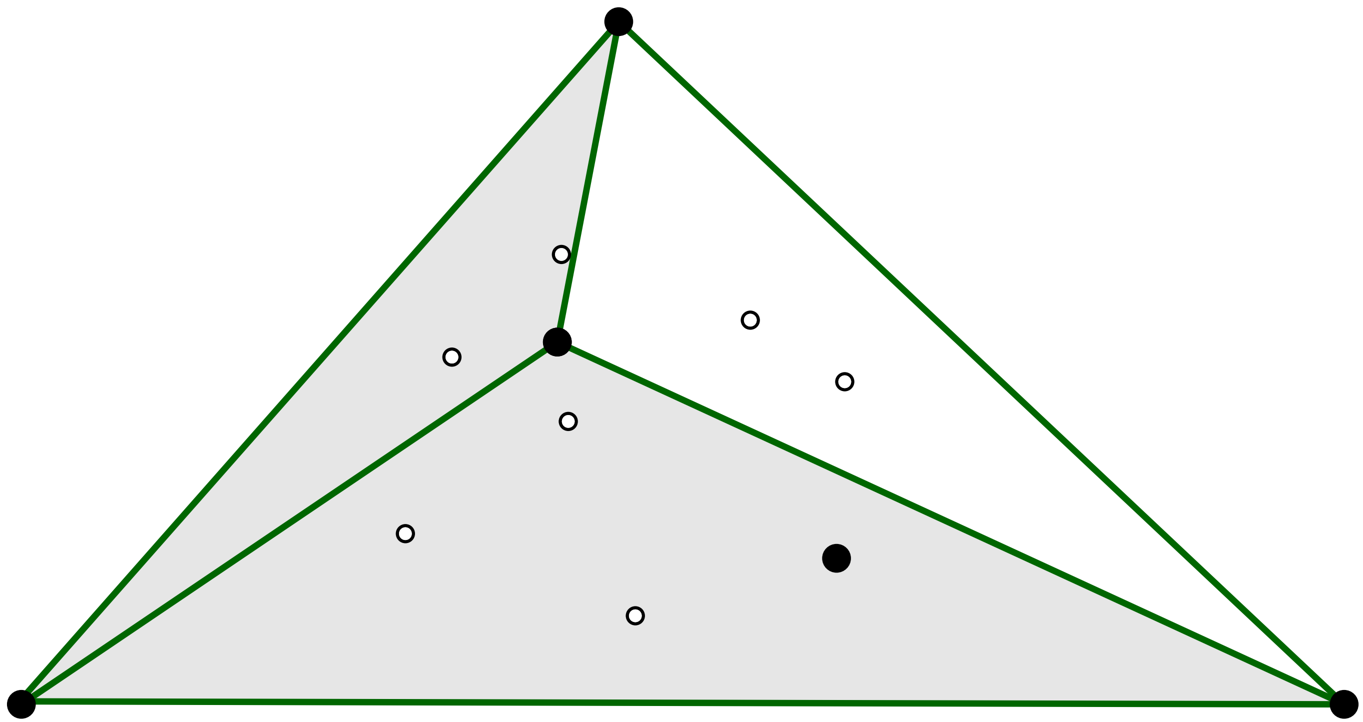

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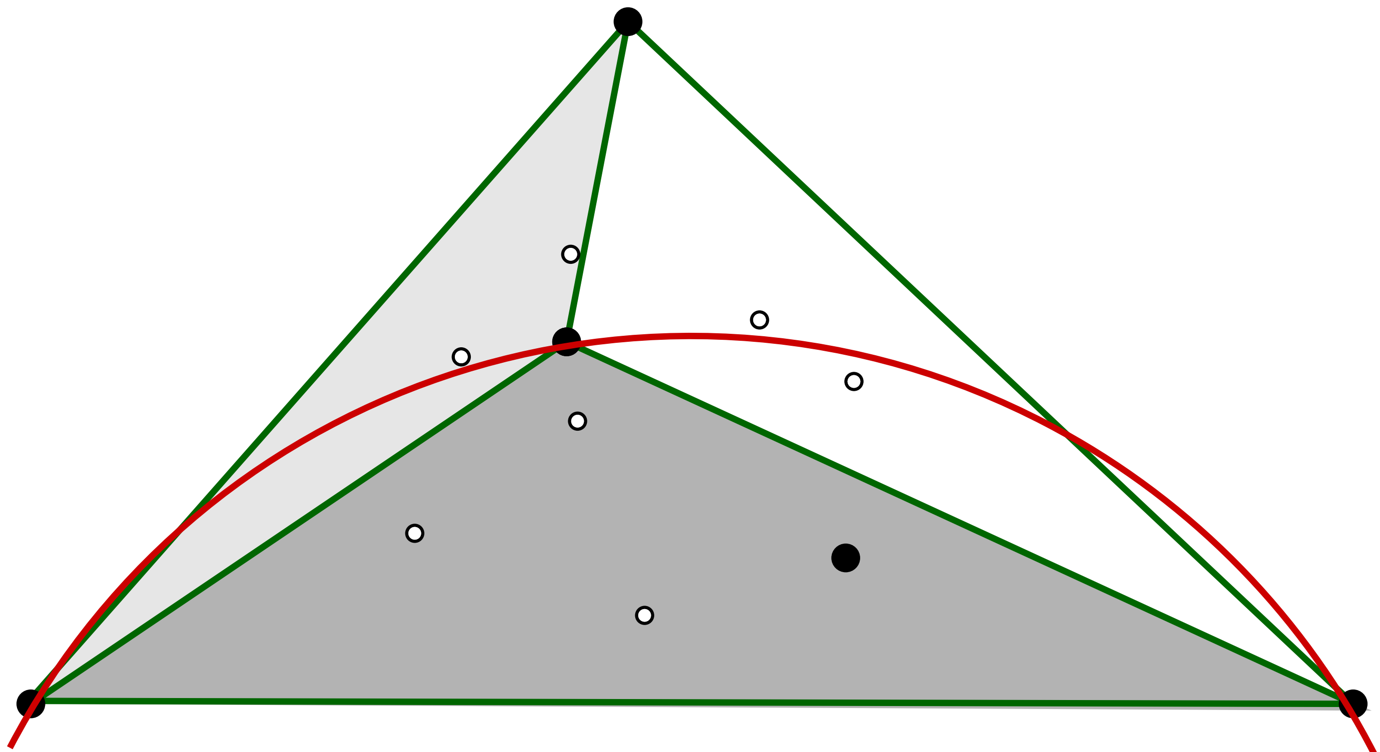

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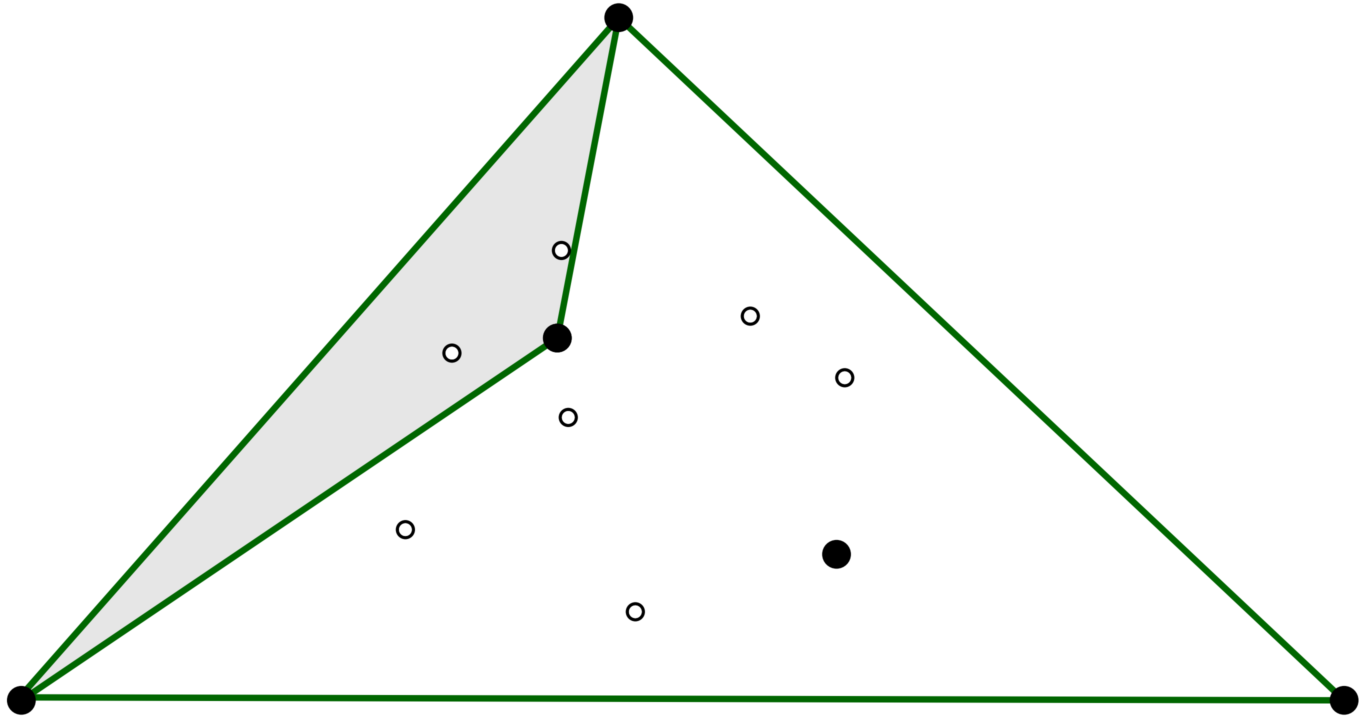

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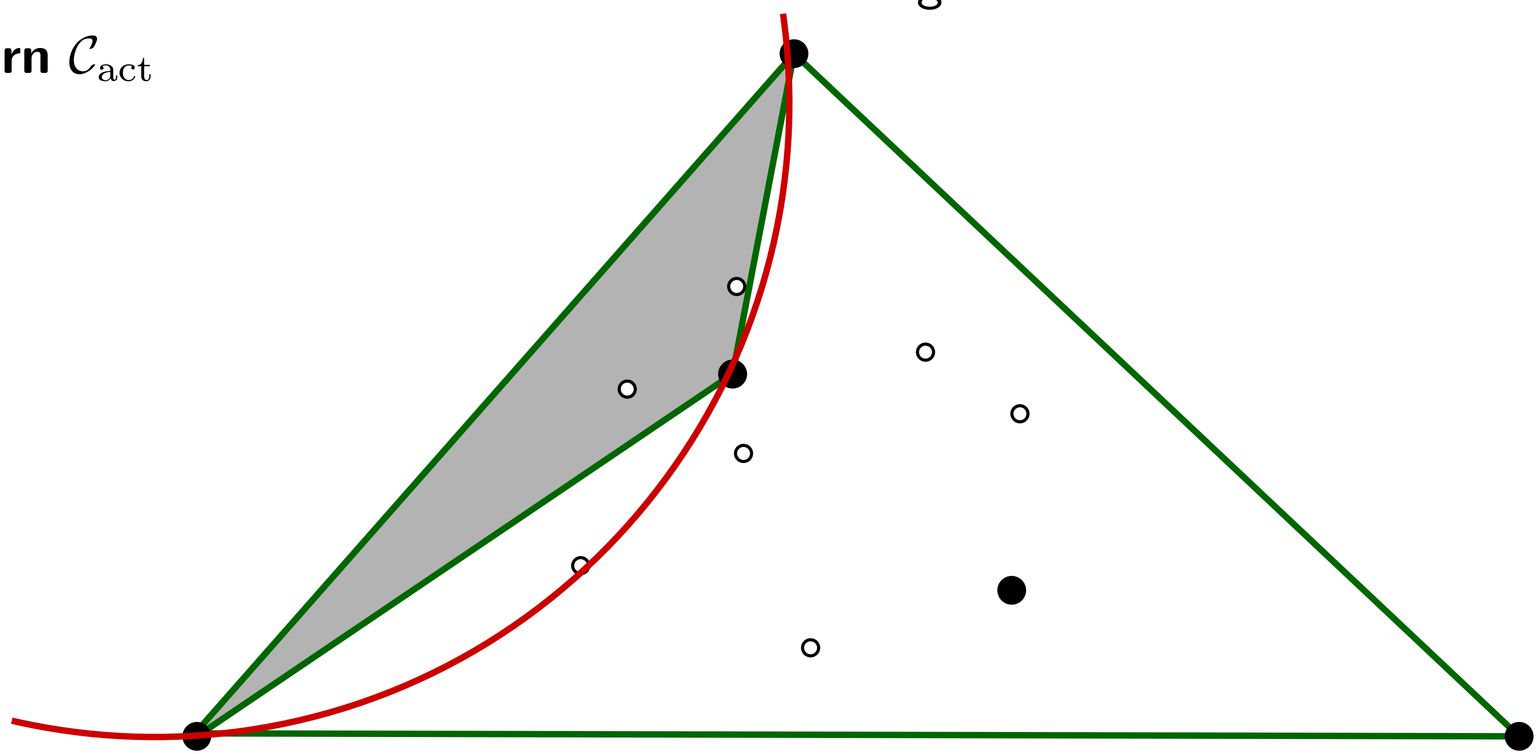

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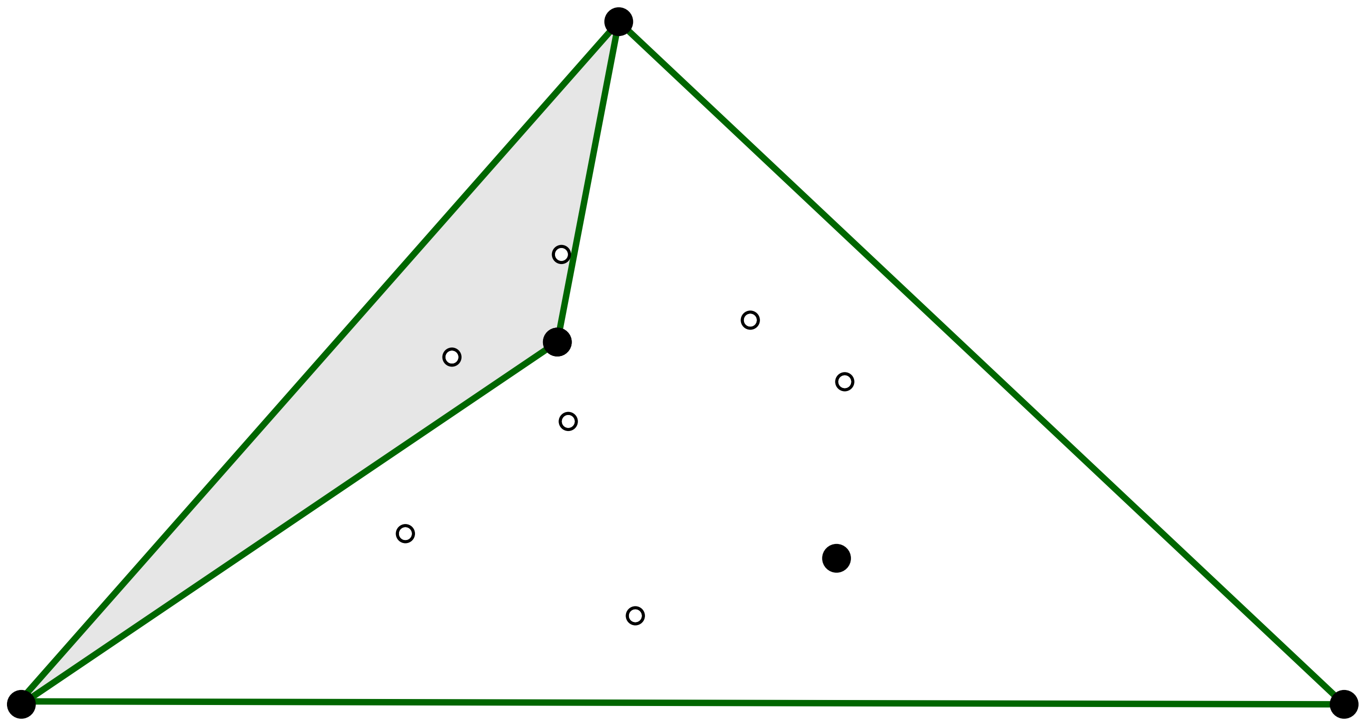

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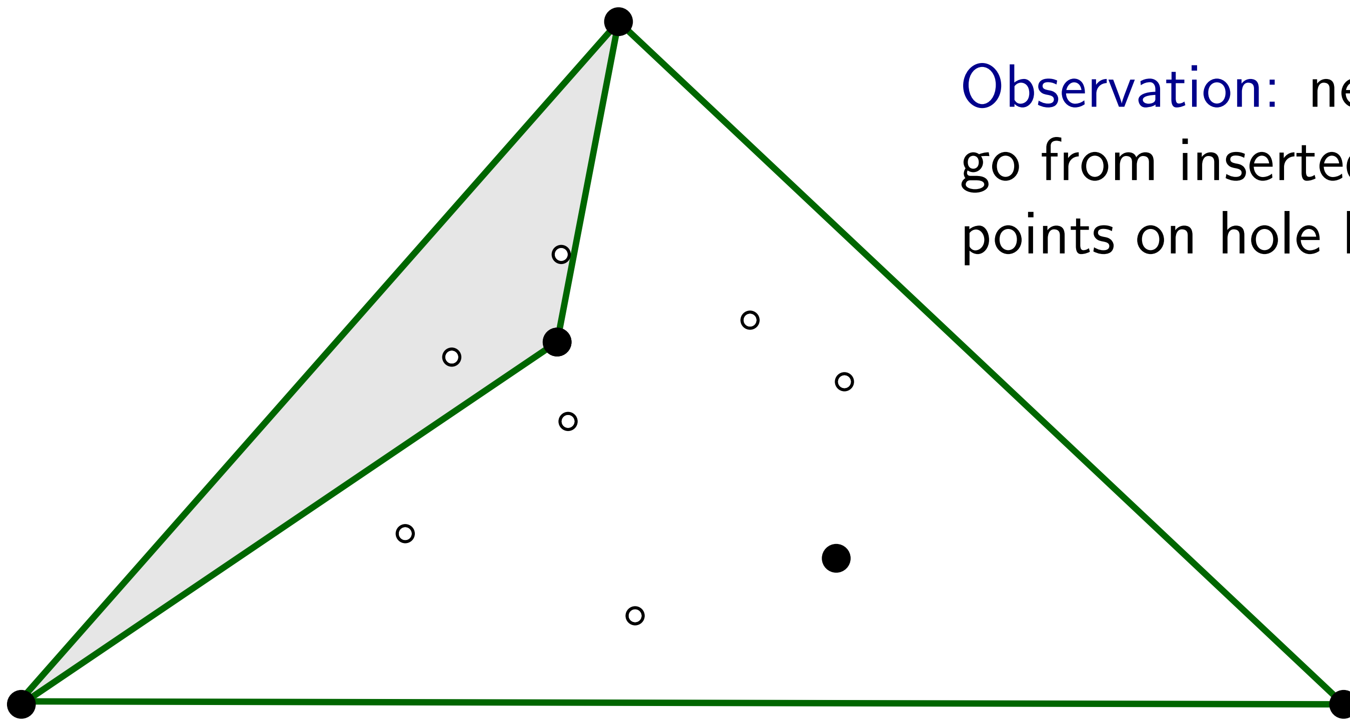

Observation: new edges go from inserted point to points on hole boundary

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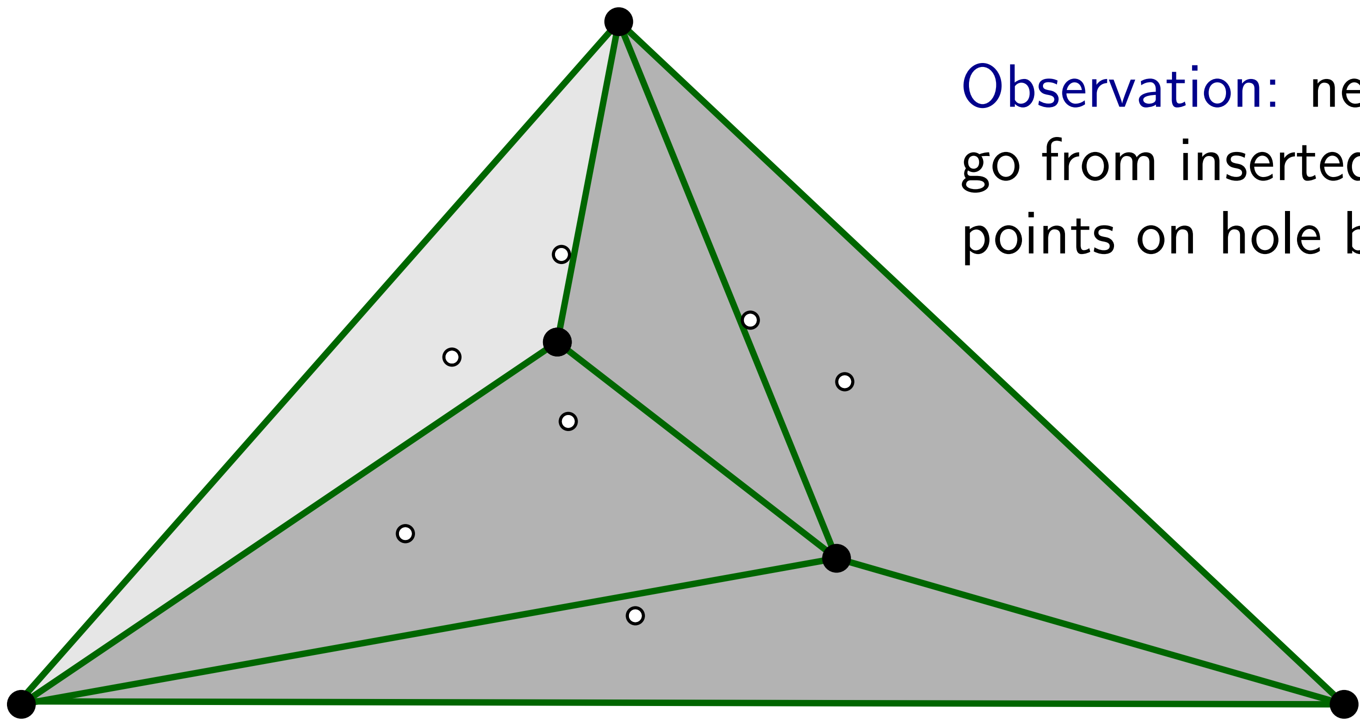

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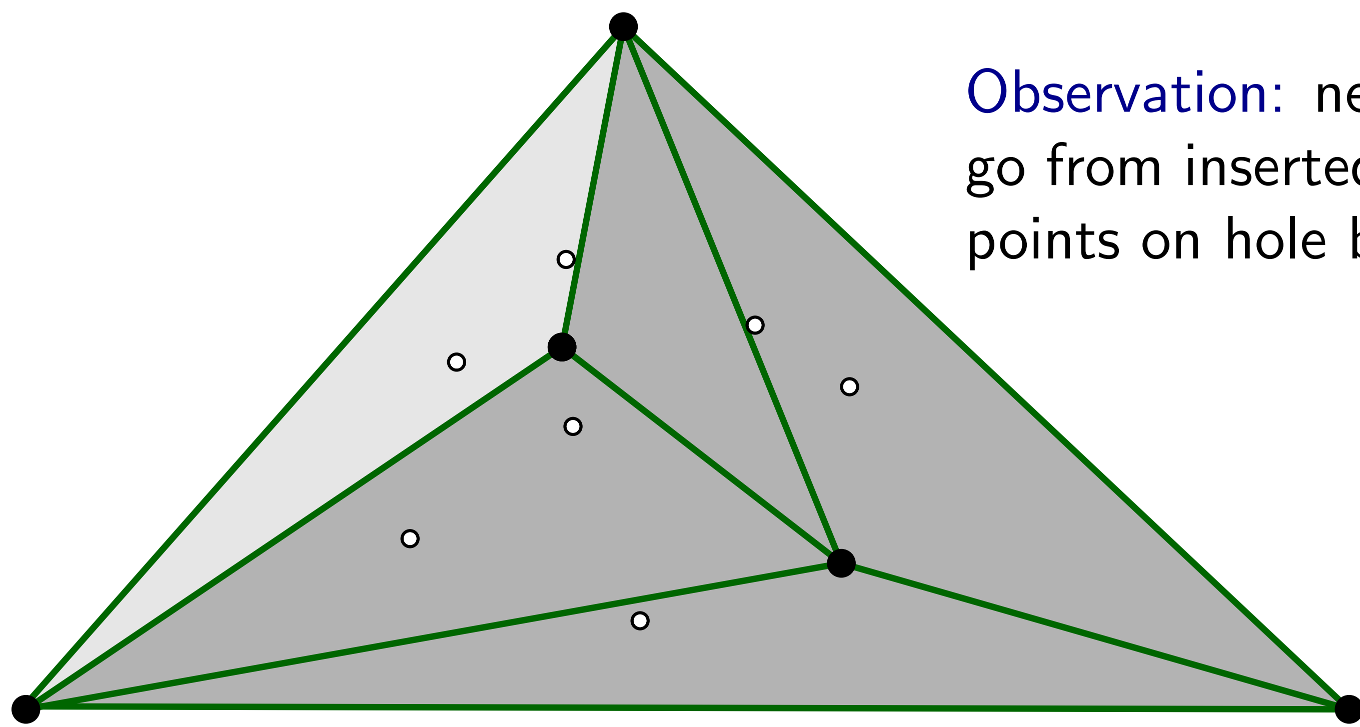
return $\mathcal{C}_{\text{act}}$



Observation: new edges go from inserted point to points on hole boundary

Theorem. Let $S_j := \{x_1, \ldots, x_j\}$. Then

(i) $\mathrm{E}\left[\,|\mathcal{C}_{\mathrm{act}}(S_j) \setminus \mathcal{C}_{\mathrm{act}}(S_{j-1})|\,\right] = O\left(\frac{\mathrm{E}[\text{size of } \mathcal{C}_{\mathrm{act}}(S_j)]}{j}\right)$

(ii) The total size of the conflict lists of the active configurations appearing over the course of the algorithm is $O\left(\sum_{j=1}^{n} \frac{n}{j^2} \cdot \mathrm{E}\left[\,|\mathcal{C}_{\mathrm{act}}(S_j)|\,\right]\right)$

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Delaunay triangulation in the plane:

size of $\mathcal{C}_{\mathrm{act}}(S_j) = \#$(triangles of Delaunay triangulation of j points) $= O(j)$

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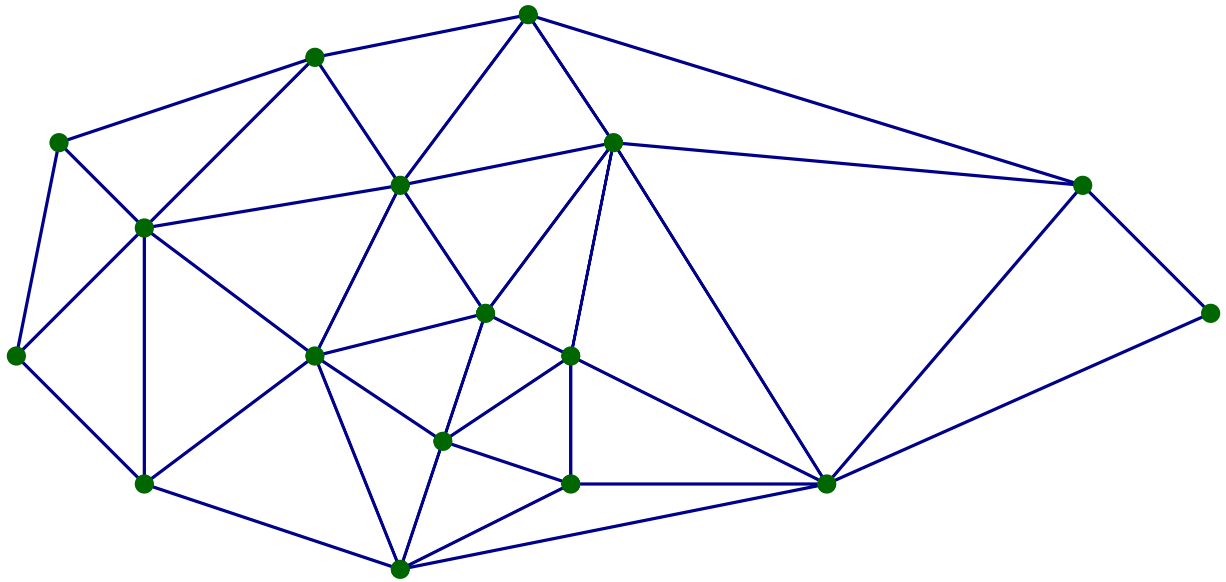
Theorem. The Delaunay triangulation of a set of n points in the plane can be computed in $O(n \log n)$ expected time, using RIC.

Voronoi Diagrams and Delaunay Triangulations

Fun Facts and Application

Voronoi Diagrams and Delaunay Triangulations: Fun Facts



dilation (= stretch factor = spanning ratio) of Delaunay triangulation is at most 1.998.

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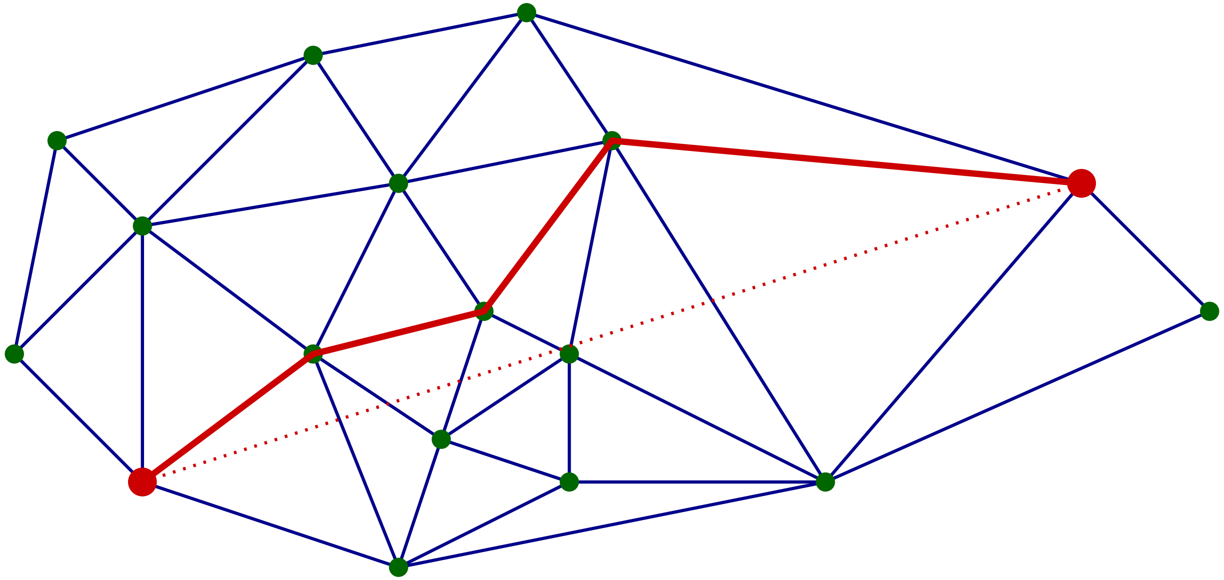

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Voronoi diagram in $\mathbb{R}^d$ $\quad\equiv\quad$ half-space intersection in $\mathbb{R}^{d+1}$ $\quad\approx\quad$ convex hull in $\mathbb{R}^{d+1}$

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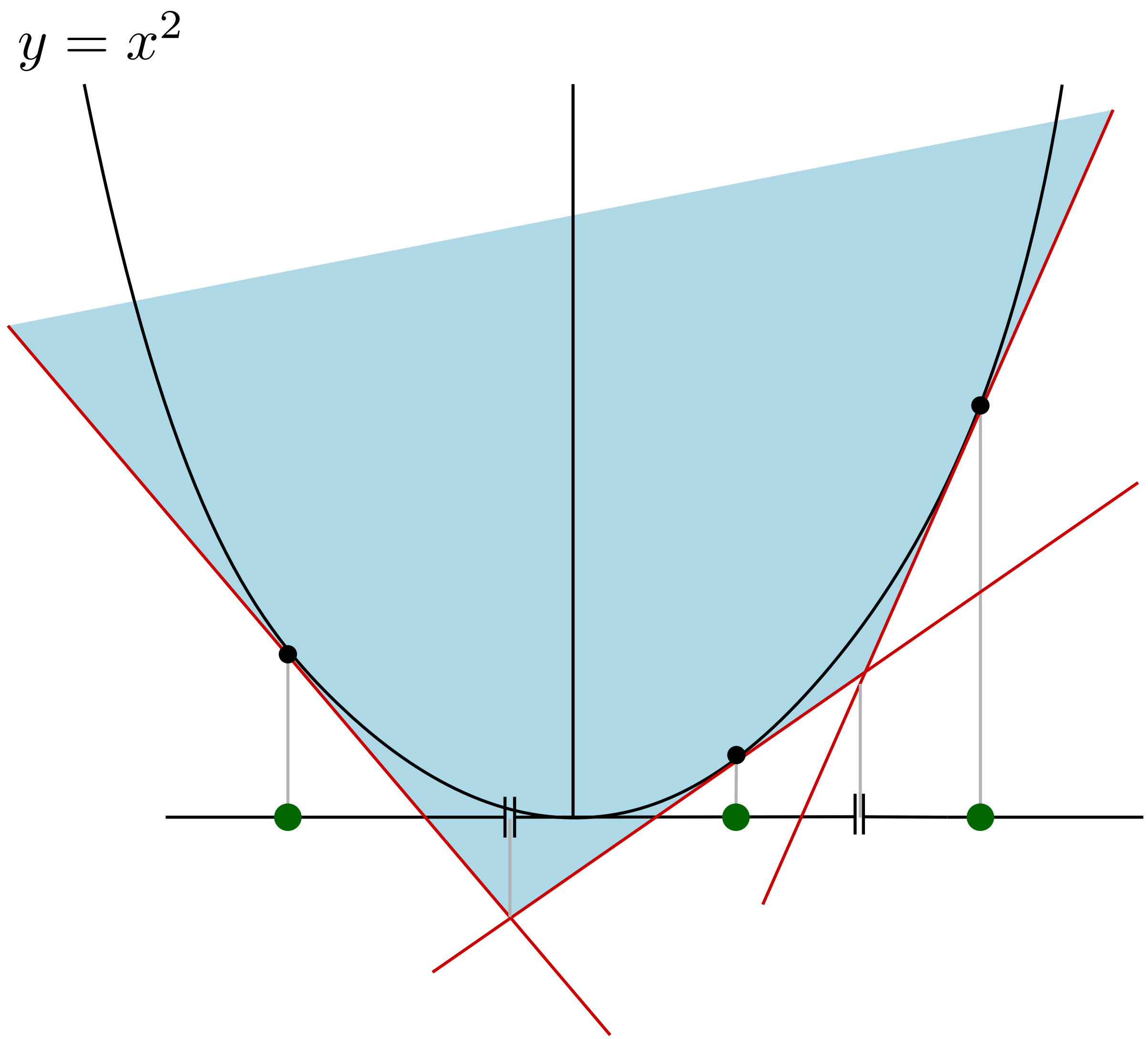

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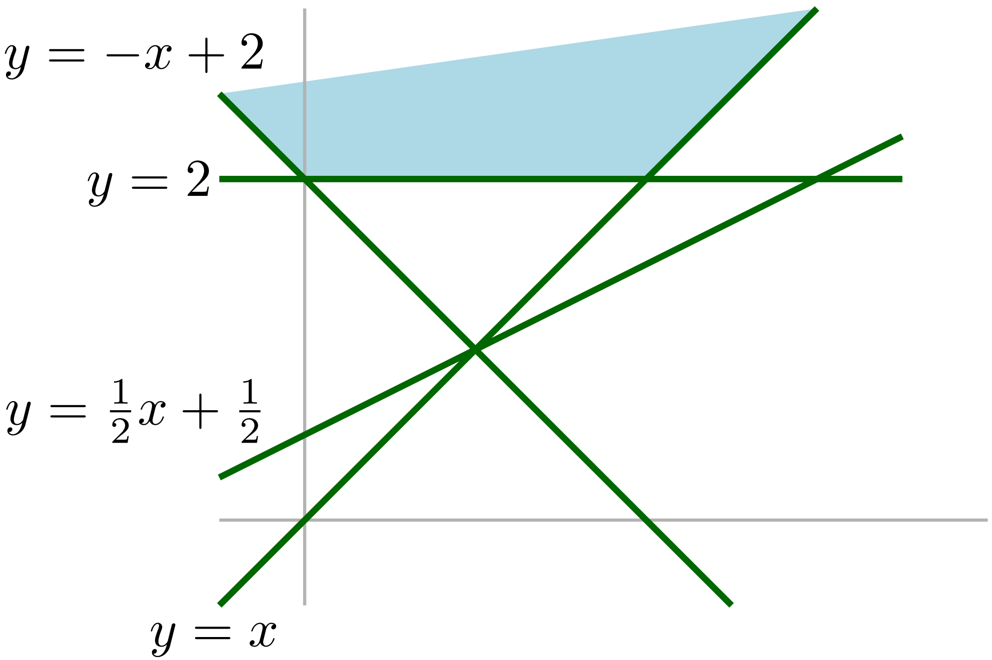
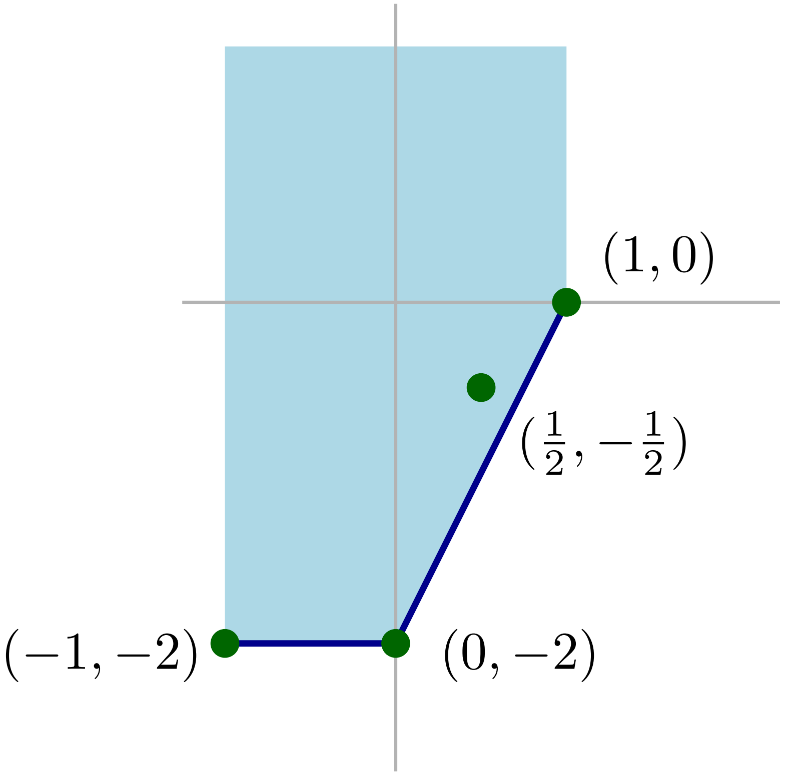
map line $y = ax + b$ to point $(a, -b)$

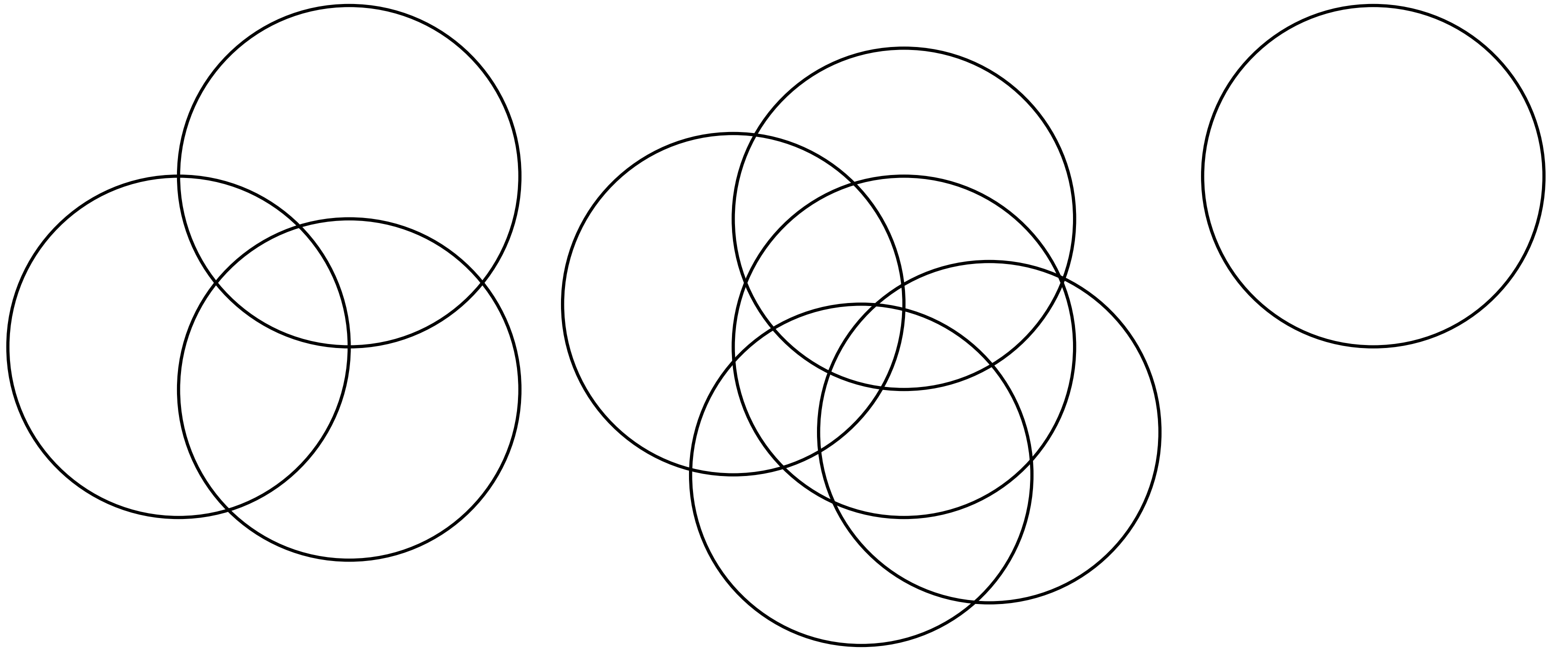


$\Longrightarrow$



upper envelope $\equiv$ lower hulll

Delaunay Triangulations: Application to CF-Coloring



for $q \in \mathbb{R}^2$ define $D(q) :=$ { disks containing q }

Conflict-free coloring: coloring of disks such that, for any q with $S(q) \neq \emptyset$, the set $D(q)$ has a disk with a unique color

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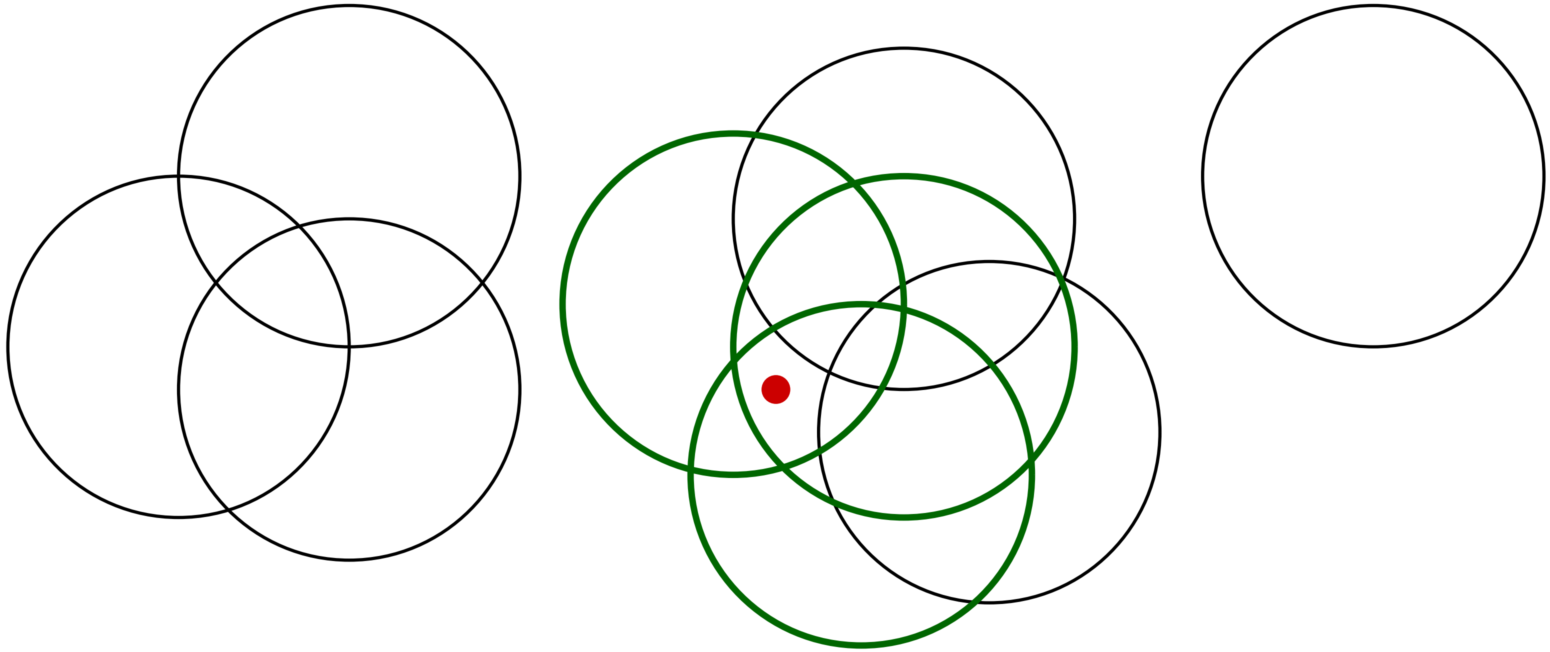

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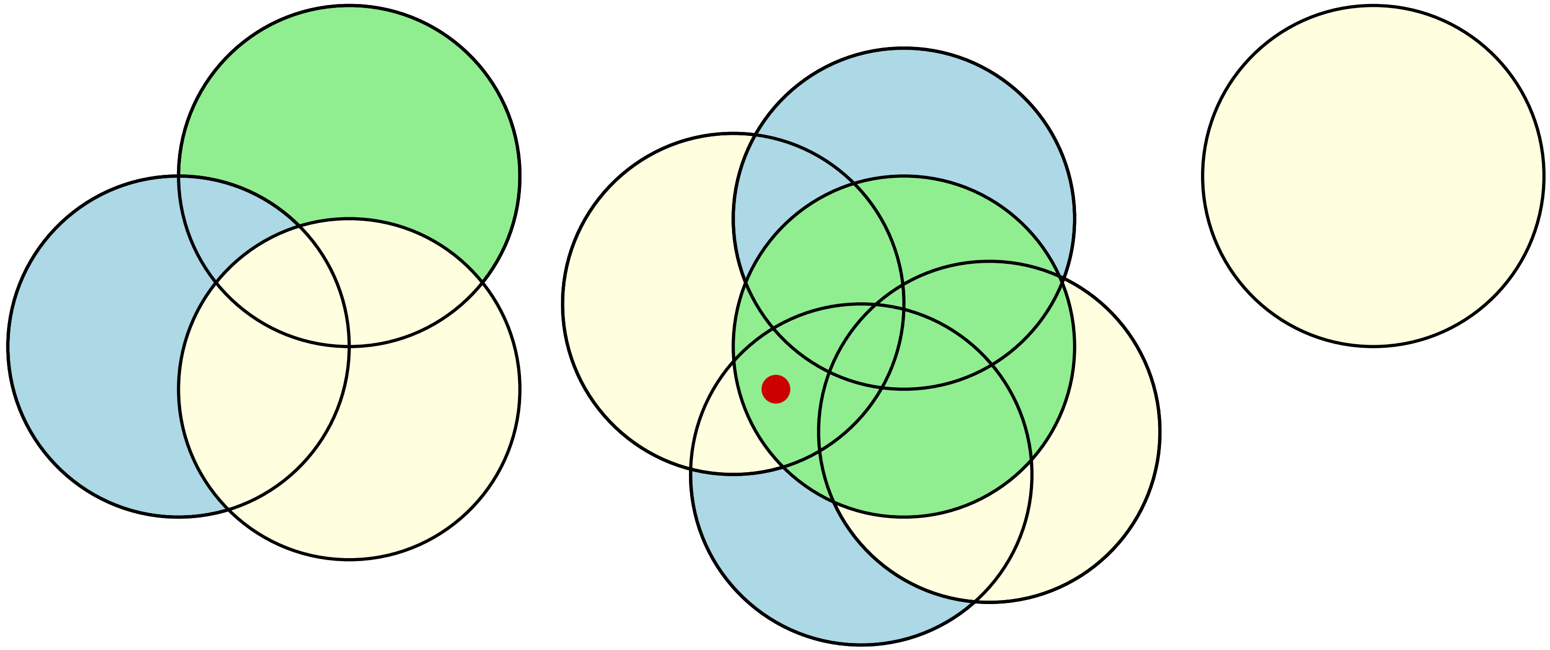

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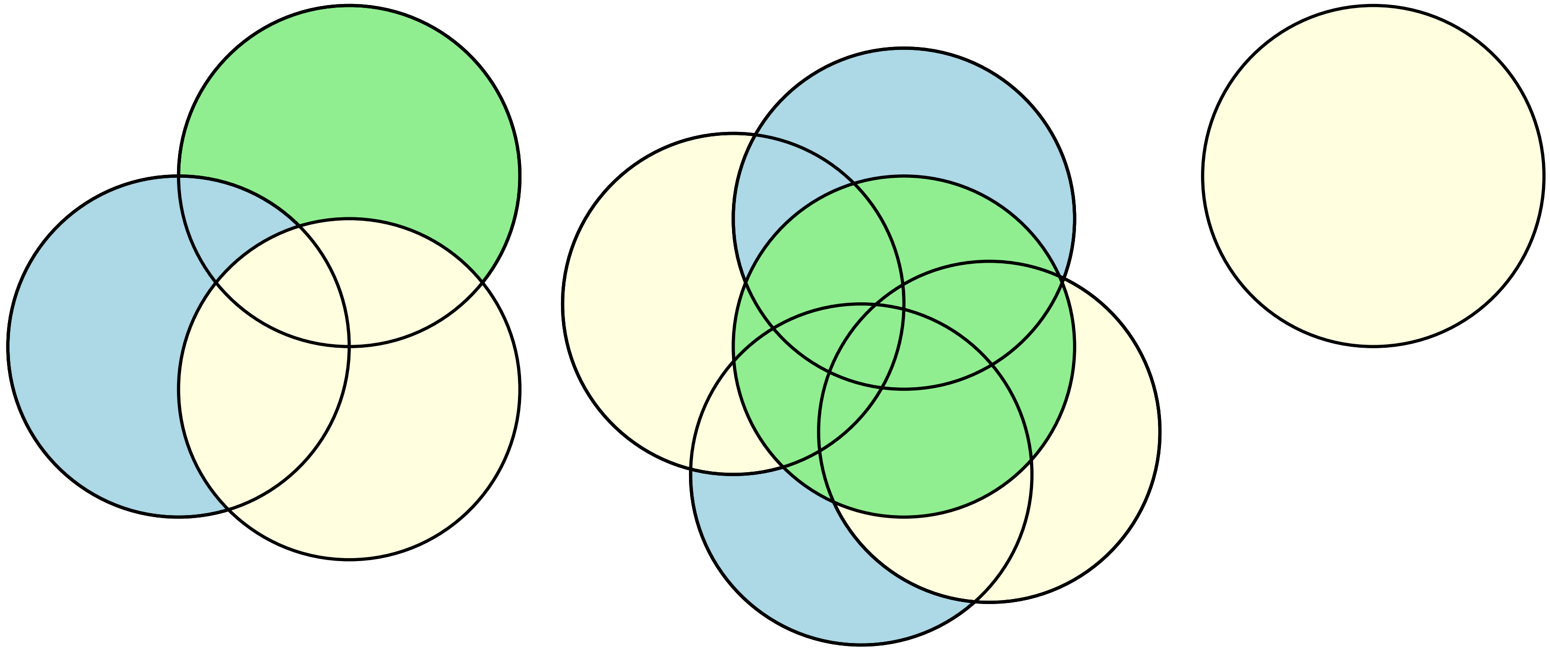


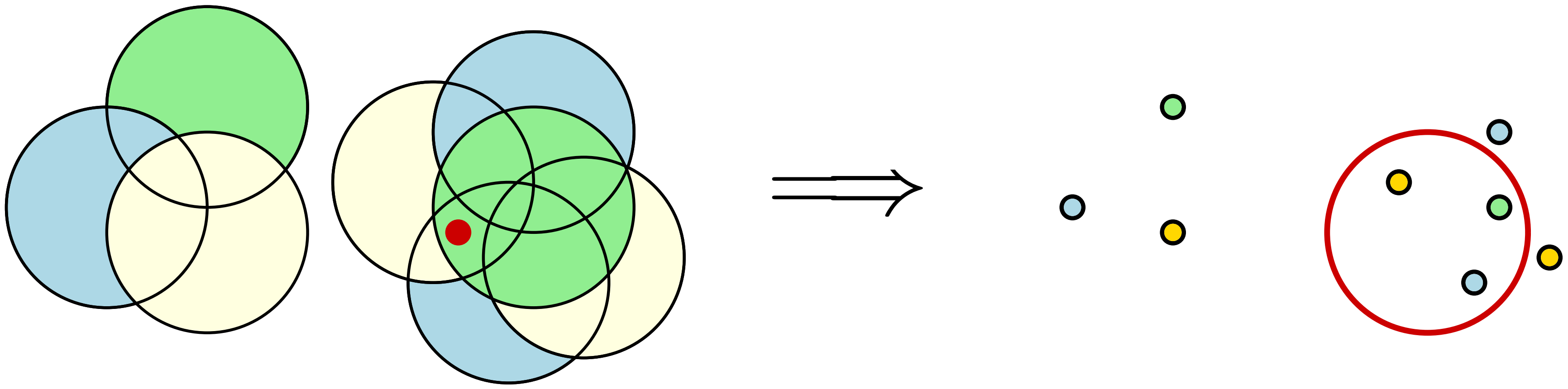
Theorem. For any set of n unit disks, there exists a conflict-free coloring with $O(\log n)$ colors, and this is best possible.

Delaunay Triangulations: Application to CF-Coloring

Invert problem: color disk centers with respect to unit disks as ranges

Delaunay Triangulations: Application to CF-Coloring

Invert problem: color disk centers with respect to unit disks as ranges



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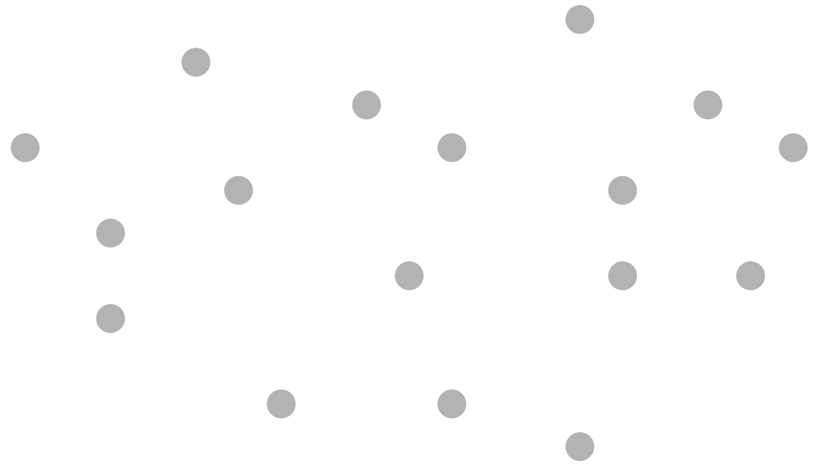

The coloring algorithm

Initally $P = \{\text{all points}\}$ and $i = 1$

1. $I :=$ max independent set in Delaunay triangulation
2. Give all points in I color i
3. Set $i := i + 1$ and recurse on $P \setminus I$

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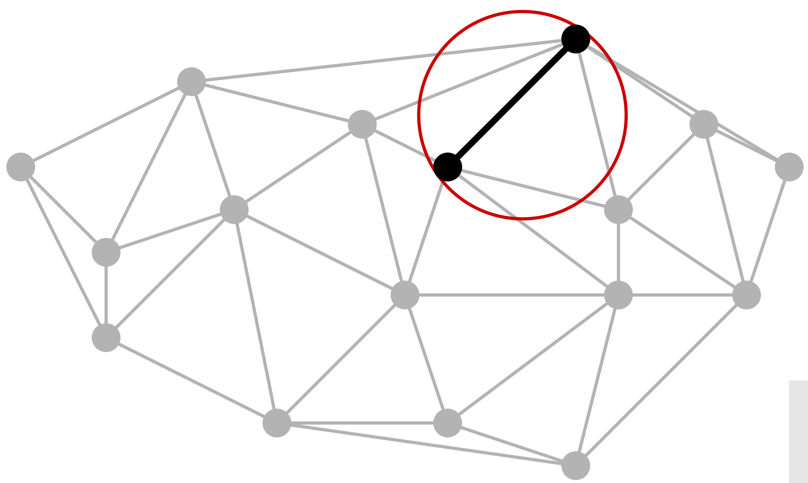

(p, q) is edge in DT iff there is a circle containing only p, q

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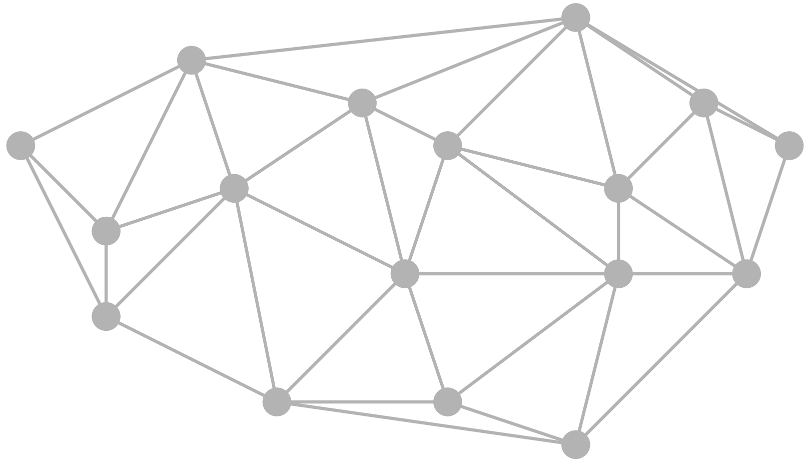

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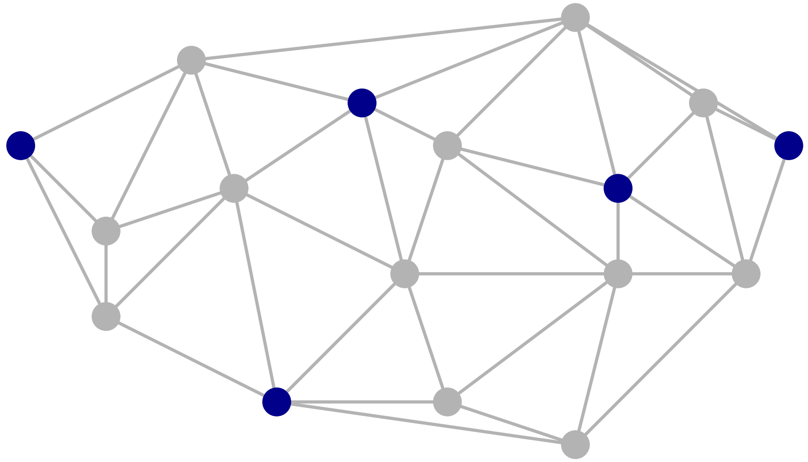

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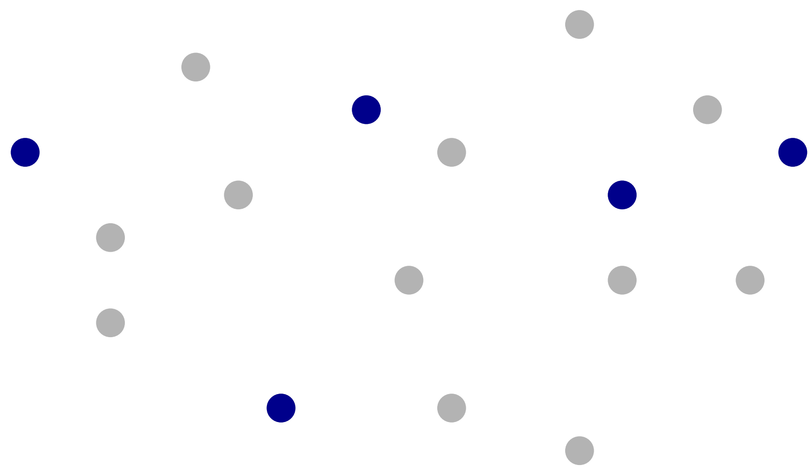

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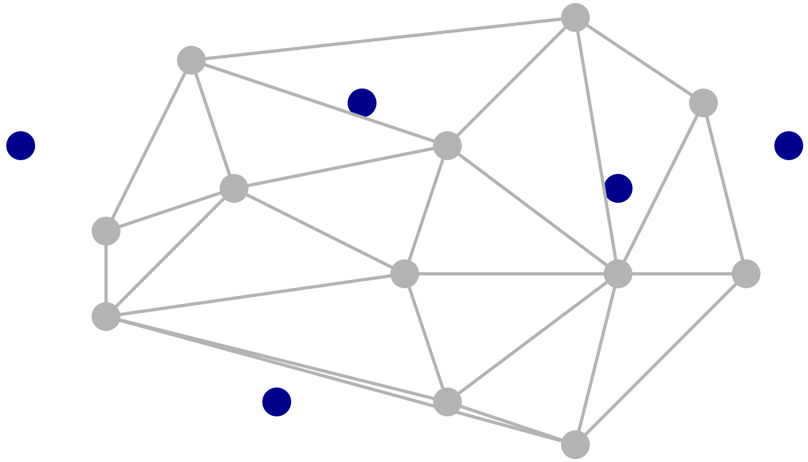

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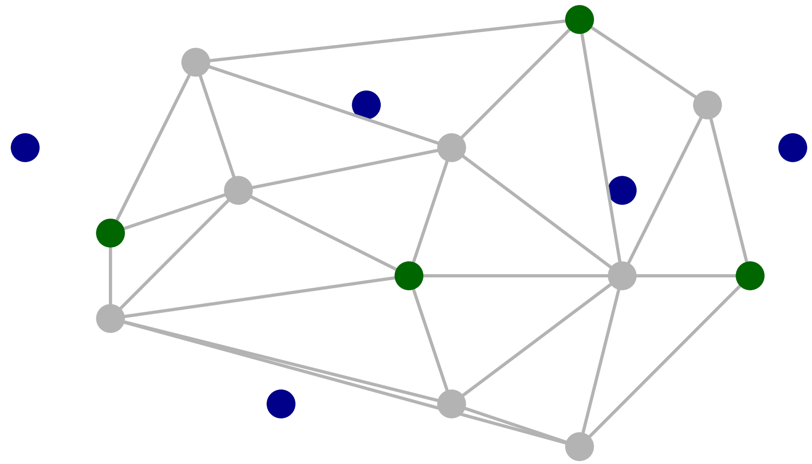

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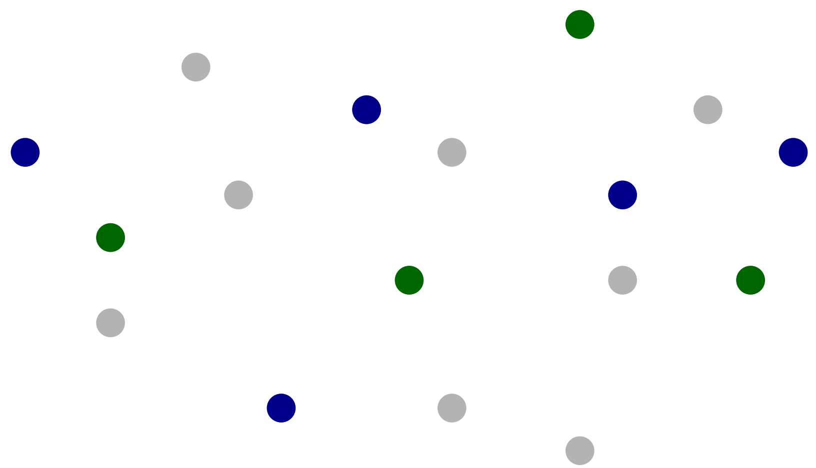

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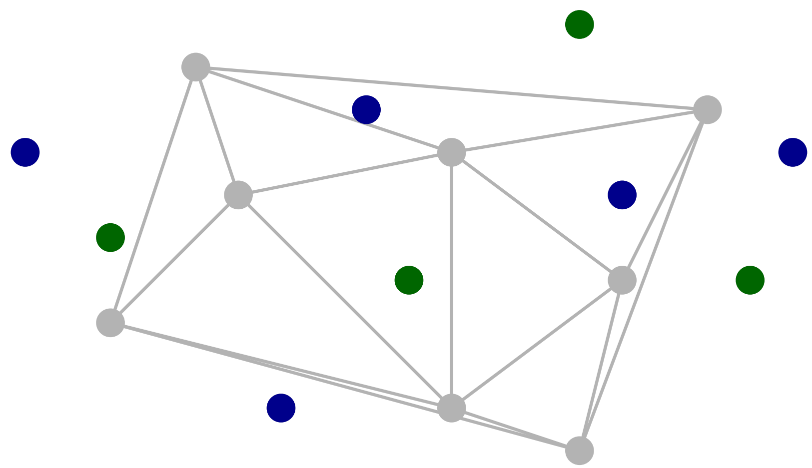

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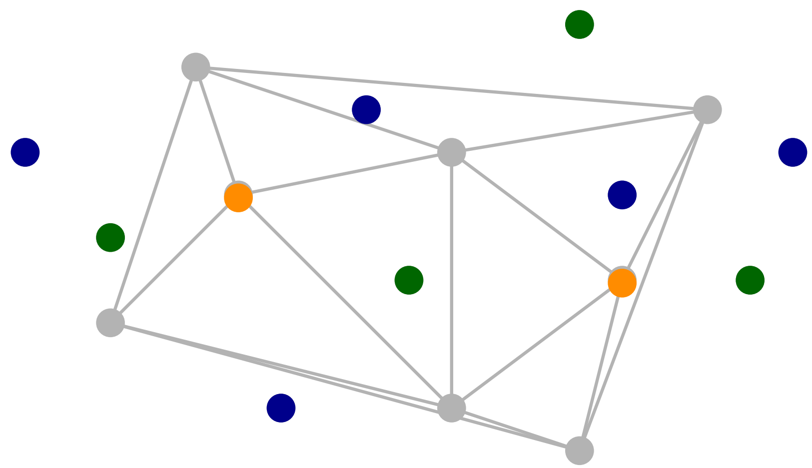

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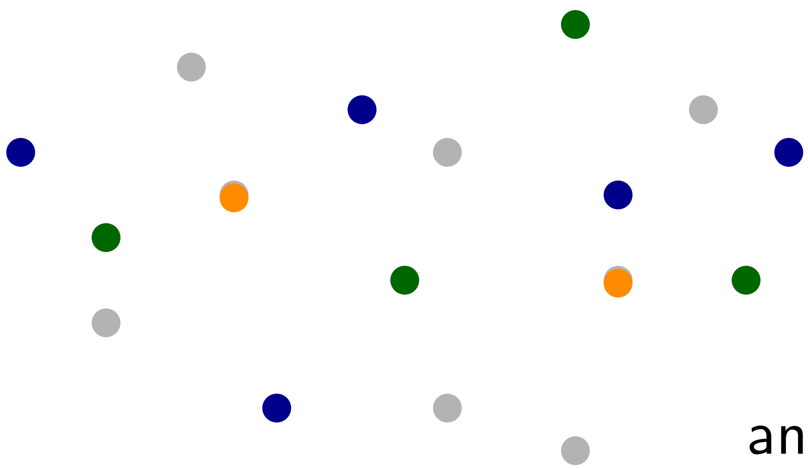

and so on . . .

The coloring algorithm

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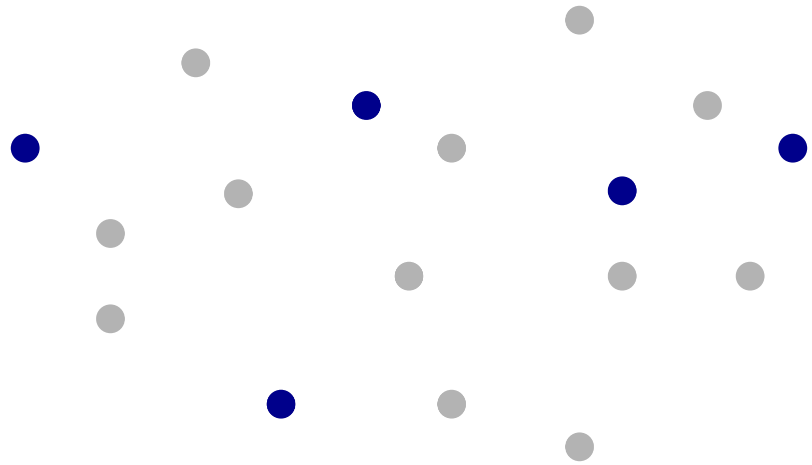
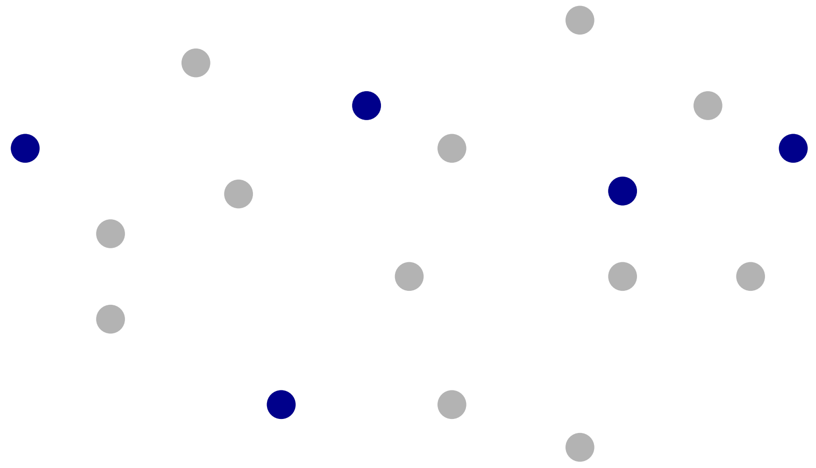


Claim. Number of colors $= O(\log n)$.

Delaunay Triangulations: Application to CF-Coloring



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- Four Color Theorem $\implies$ size max indep set $\geqslant \frac{1}{4}n$

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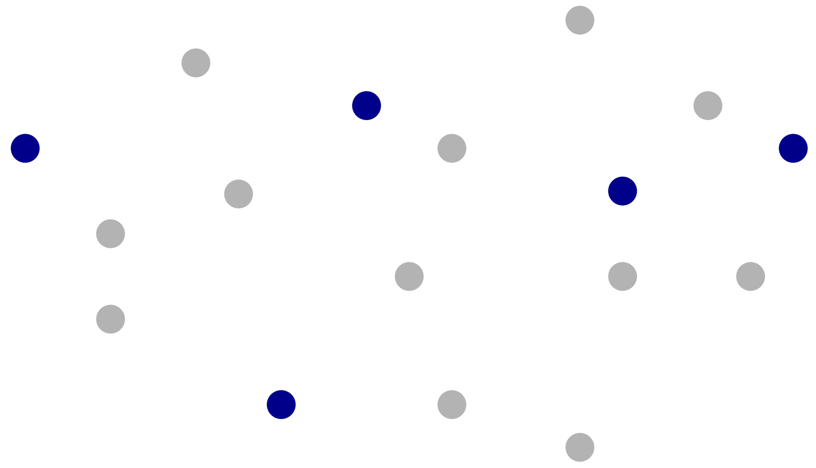
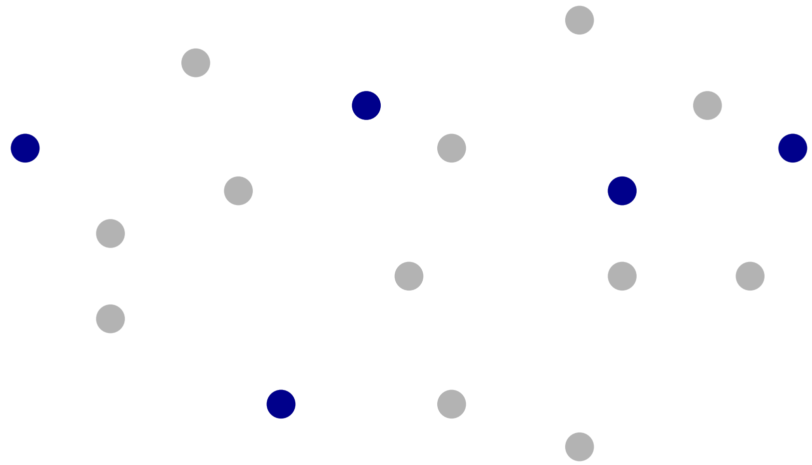


Claim. Number of colors $= O(\log n)$.

- Four Color Theorem $\implies$ size max indep set $\geqslant \frac{1}{4}n$
- $C(n) :=$ number of colors

$$C(n) \leqslant 1 + C\left(\frac{3}{4}n\right) \implies C(n) = O(\log n)$$

Delaunay Triangulations: Application to CF-Coloring



Claim. Coloring is conflict-free.

any non-empty disk must have point with unique color

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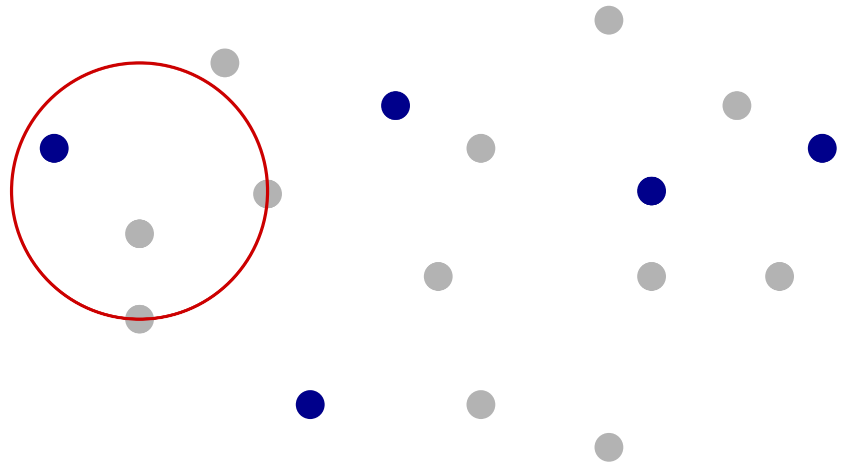

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- disk has single point with color 1

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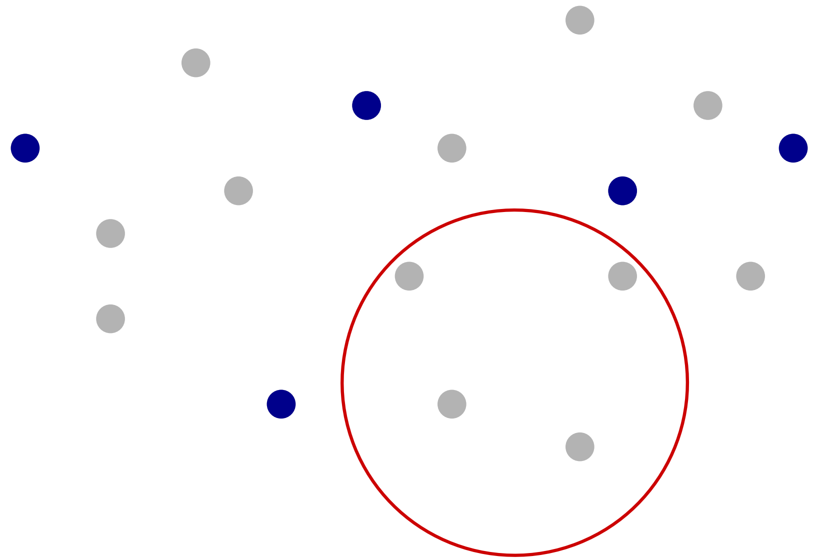

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- disk has single point with color 1
- disk has no point with color 1: induction

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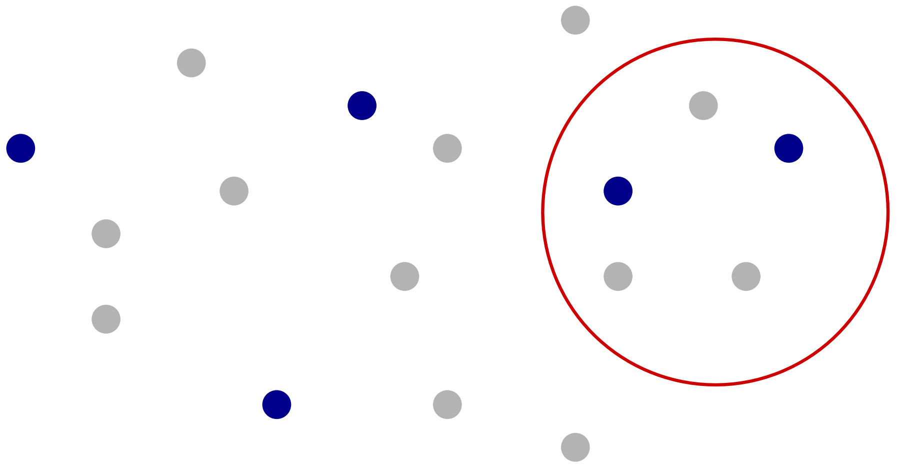
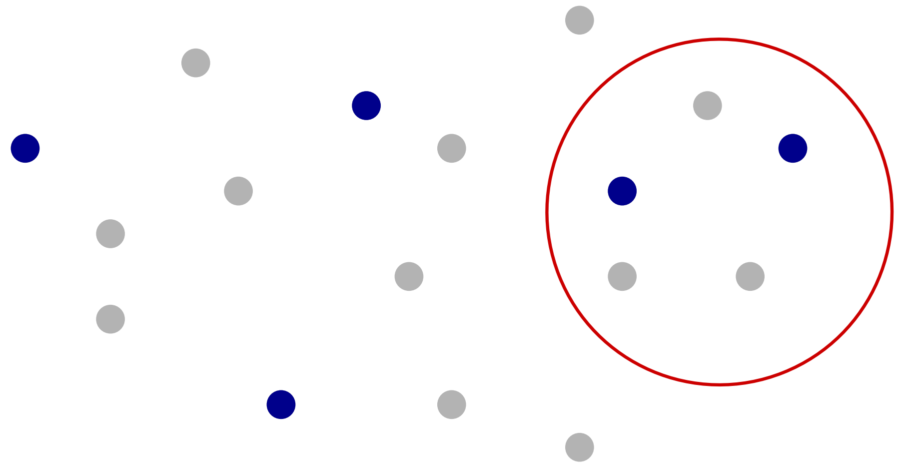


Claim. Coloring is conflict-free.

any non-empty disk must have point with unique color

- disk has single point with color 1
- disk has no point with color 1
- disk has $\geqslant 2$ points with color 1
  disk must contain other points $\Longrightarrow$ induction



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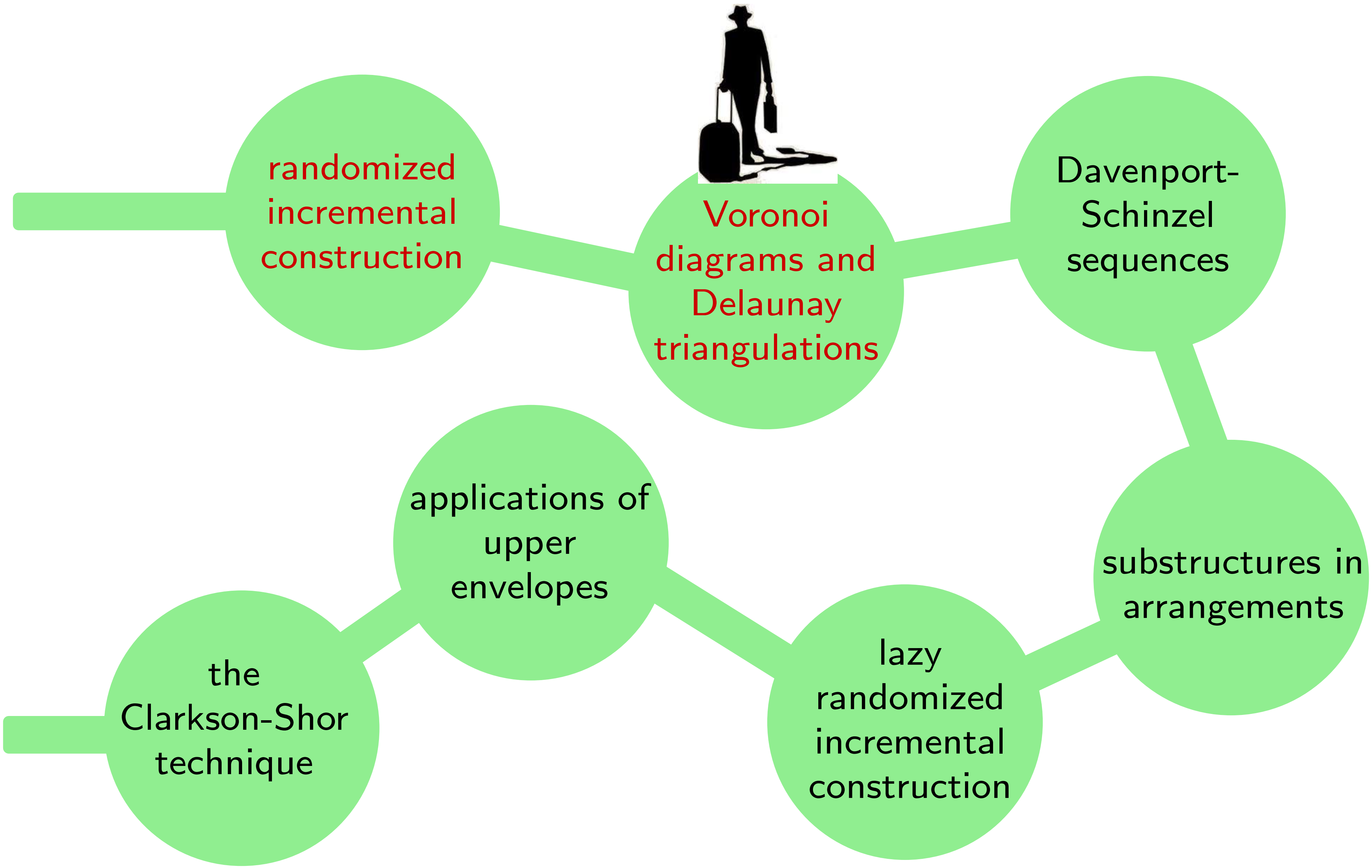

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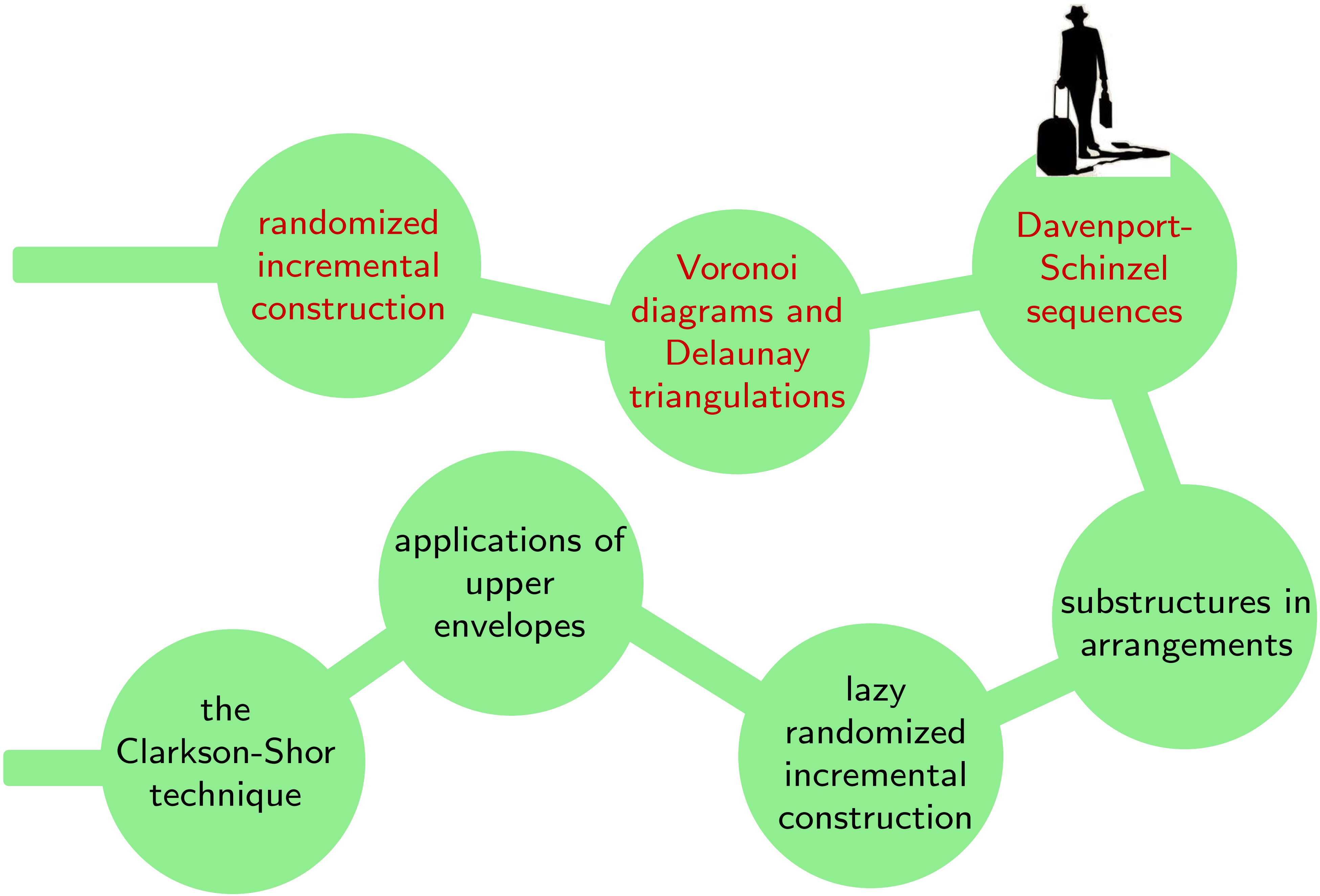

Davenport-Schinzel sequences

A COMBINATORIAL PROBLEM CONNECTED WITH DIFFERENTIAL EQUATIONS.

By H. Davenport and A. Schinzel.

1. Let

(1) $$F(D)f(x) = 0$$

be a (homogeneous) linear differential equation with constant coefficients, of order d. Suppose that $F(D)$ has real coefficients, and that the roots of $F(\lambda) = 0$ are all real though not necessarily distinct. As is well known, any solution of (1) is of the form

(2) $$f(x) = P_1(x)e^{\lambda_1 x} + \cdots + P_k(x)e^{\lambda_k x},$$

where $\lambda_1, \cdots, \lambda_k$ are the distinct roots of $F(\lambda) = 0$ and $P_1(x), \cdots, P_k(x)$ are polynomials of degrees at most $m_1 - 1, \cdots, m_k - 1$, where $m_1, \cdots, m_k$ are the multiplicities of the roots, so that $m_1 + \cdots + m_k = d$.

Let

(3) $$f_1(x), \cdots, f_n(x)$$

be n distinct (but not necessarily independent) solutions of (1). For each real number x, apart from a finite number of exceptions, there will be just one of the functions (3) which is greater than all the others. We can therefore dissect the real line into N intervals

$$(-\infty, x_1), (x_1, x_2), \cdots, (x_{N-1}, \infty)$$

such that inside any one of the intervals (x_{j-1}, x_j) a particular one of the functions (3) is the greatest, and such that this function is not the same for two consecutive intervals. It is almost obvious that N is finite, and a formal proof will be given below.

The problem of finding how large N can be, for given d and given n, was proposed to one of us (in a slightly different form) by K. Malanowski. This problem can be made to depend on a purely combinatorial problem, by the following considerations. With each $j = 1, 2, \cdots, N$ there is associated the integer $i = i(j)$ for which $f_i(x)$ is the greatest of the functions (3) in the interval (x_{j-1}, x_j). (We write $x_0 = -\infty$ and $x_N = \infty$ for convenience.) This defines a sequence of N terms

(4) $$i(1), i(2), \cdots, i(N),$$

Received August 26, 1964.
684

American Journal of Mathematics 87:684–694 (1965)

Harold Davenport (1907–1965)

Andrzej Schinzel (1937–2021)

A combinatorial problem

Consider a sequence over the alphabet $\{1, \ldots, n\}$ such that

- $\ldots\ i\ i\ \ldots$ does not appear
- $\ldots\underbrace{i\ldots j\ldots i\ldots j}_{s+2\text{ times}}\ldots$ does not appear

How long can such a sequence be?

Davenport-Schinzel sequence of order s (over alphabet of size n) is sequence that does not contain the following:

- $\ldots\ i\ i\ \ldots$ — no two consecutive symbols are the same
- $\ldots\underbrace{i\ldots j\ldots i\ldots j}_{s+2\text{ times}}\ldots$ — no alternating subsequence of length $s+2$

Example $(n = 9,\ s = 2)$

- 6, 4, 5, 6, 1, 2, 2, 7, 3
- 2, 5, 1, 2, 7, 8, 7, 1, 3, 4
- 3, 6, 4, 2, 5, 1, 5, 9, 8, 9, 7

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Example $(n=9,\ s=2)$

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- 2, 5, 1, 2, 7, 8, 7, 1, 3, 4 ✕
- 3, 6, 4, 2, 5, 1, 5, 9, 8, 9, 7 V

Exercise: Determine the maximal possible length of a DS-sequence of order s as a function of n, for $s=1$, $s=2$, $s=3$, ...

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$\mathrm{DS}_s(n) :=$ maximum length of DS-sequence of order s on n symbols

- $s = 1$:

- $s = 2$:

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$\mathrm{DS}_s(n) :=$ maximum length of DS-sequence of order s on n symbols

- $s = 1$: $\left.\begin{array}{l}\text{possible sequence: } 1, 2, 3, \ldots, n \\ \text{no symbol can appear twice}\end{array}\right\} \Longrightarrow \quad \mathrm{DS}_1(n) = n$

- $s = 2$:

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- $s = 1$: $\left.\begin{array}{l}\text{possible sequence: } 1, 2, 3, \ldots, n \\ \text{no symbol can appear twice}\end{array}\right\} \implies \mathrm{DS}_1(n) = n$

- $s = 2$: possible sequence $1, 2, \ldots, n-1, n, n-1, \ldots, 2, 1$

  $\implies \mathrm{DS}_2(n) \geqslant 2n - 1$

  Proof by induction, remove symbol whose first occurrence is last, plus at most one adjacent symbol:

  $\mathrm{DS}_2(n) \leqslant \mathrm{DS}(n-1) + 2 \implies \mathrm{DS}_2(n) \leqslant 2n - 1$

Theorem. $\mathrm{DS}_s(n)$ is near-linear for any constant s. In particular,

- $\mathrm{DS}_1(n) = n$
- $\mathrm{DS}_2(n) = 2n - 1$
- $\mathrm{DS}_3(n) = \Theta(n\alpha(n))$
- $\mathrm{DS}_s(n) = o(n \log^* n)$ for any fixed constant s

where $\alpha(n)$ is the inverse Ackermann function

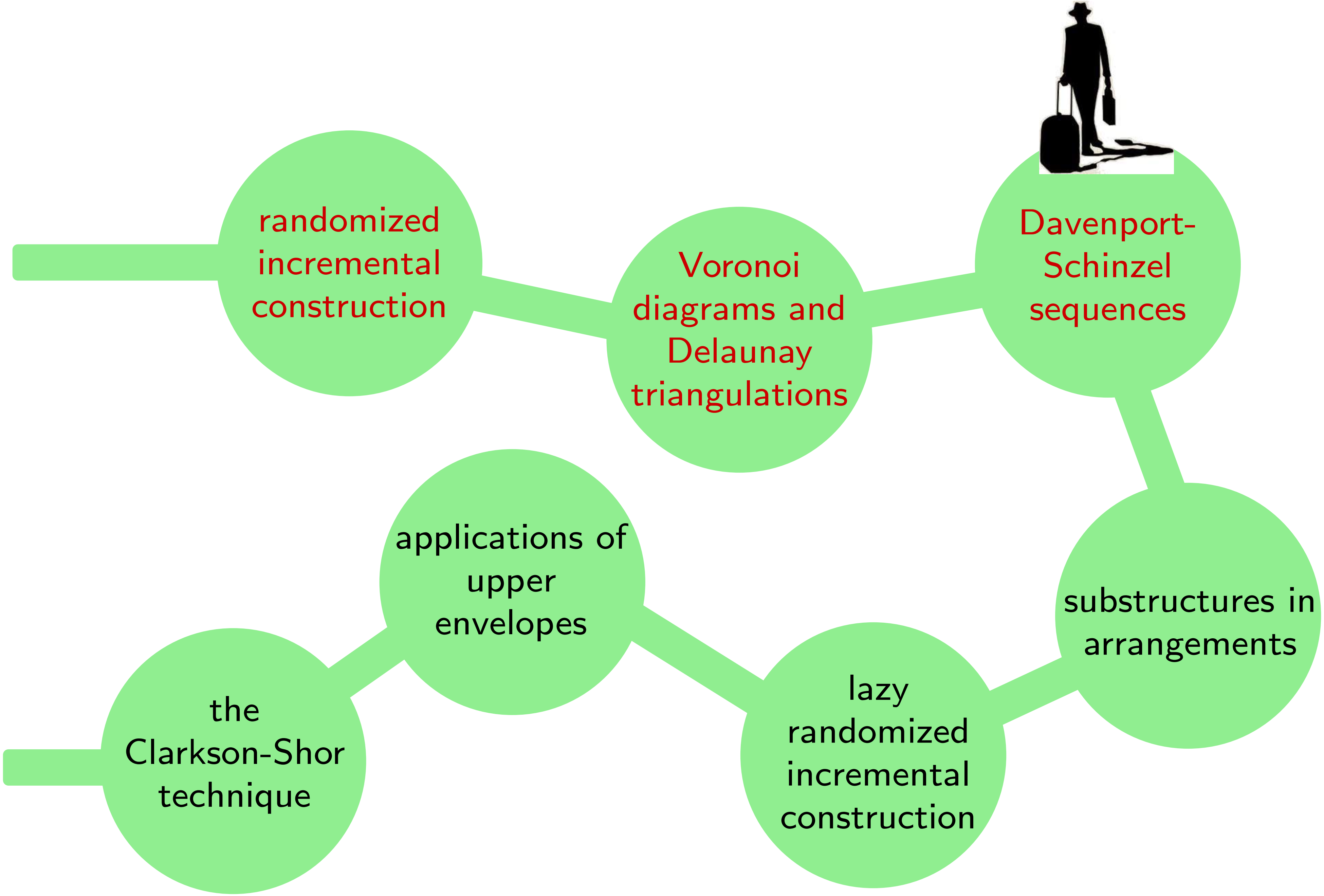
$\alpha(n)$ grows slower than super-super-super-super-super-slowly ...

$\alpha(n)$ is inverse of Ackermann function $A(n)$, where $A(n) = A_n(n)$ with:

$$\begin{array}{rcll} A_1(n) & = & 2n & \text{for } n \geqslant 1 \\ A_k(1) & = & 2 & \text{for } k \geqslant 1 \\ A_k(n) & = & A_{k-1}(A_k(n-1)) & \text{for } k \geqslant 2 \text{ and } n \geqslant 2 \end{array}$$

$A(1) = 2$, $A(2) = 4$, $A(3) = 16$, $A(4) =$ tower of 65536 2's

Course Overview



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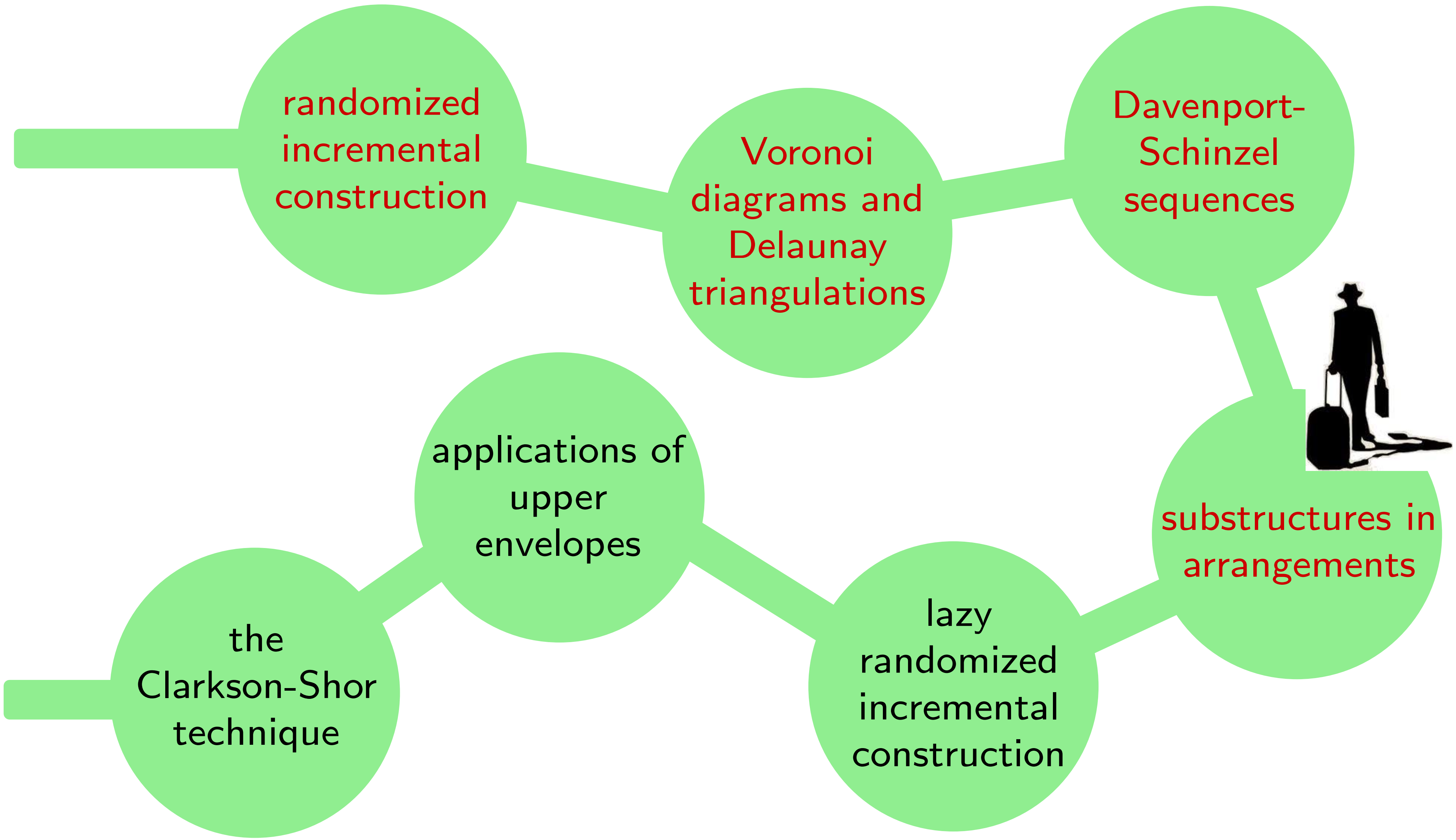
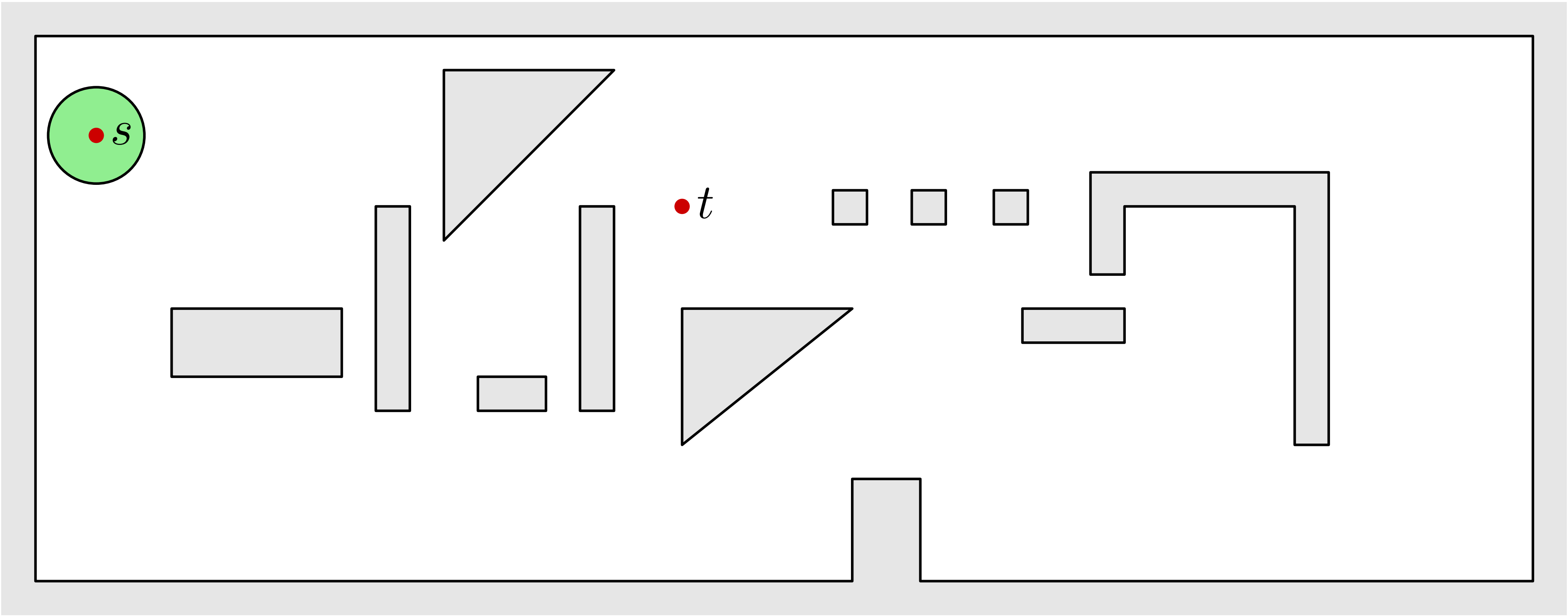
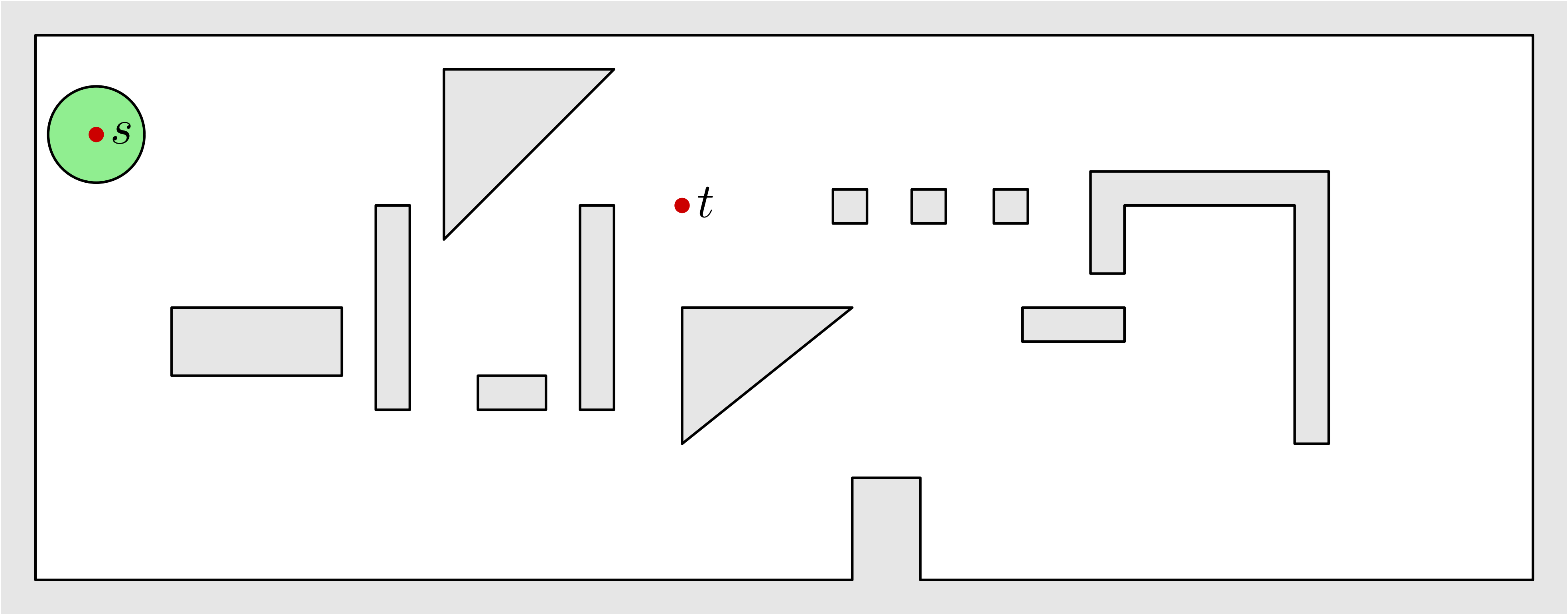
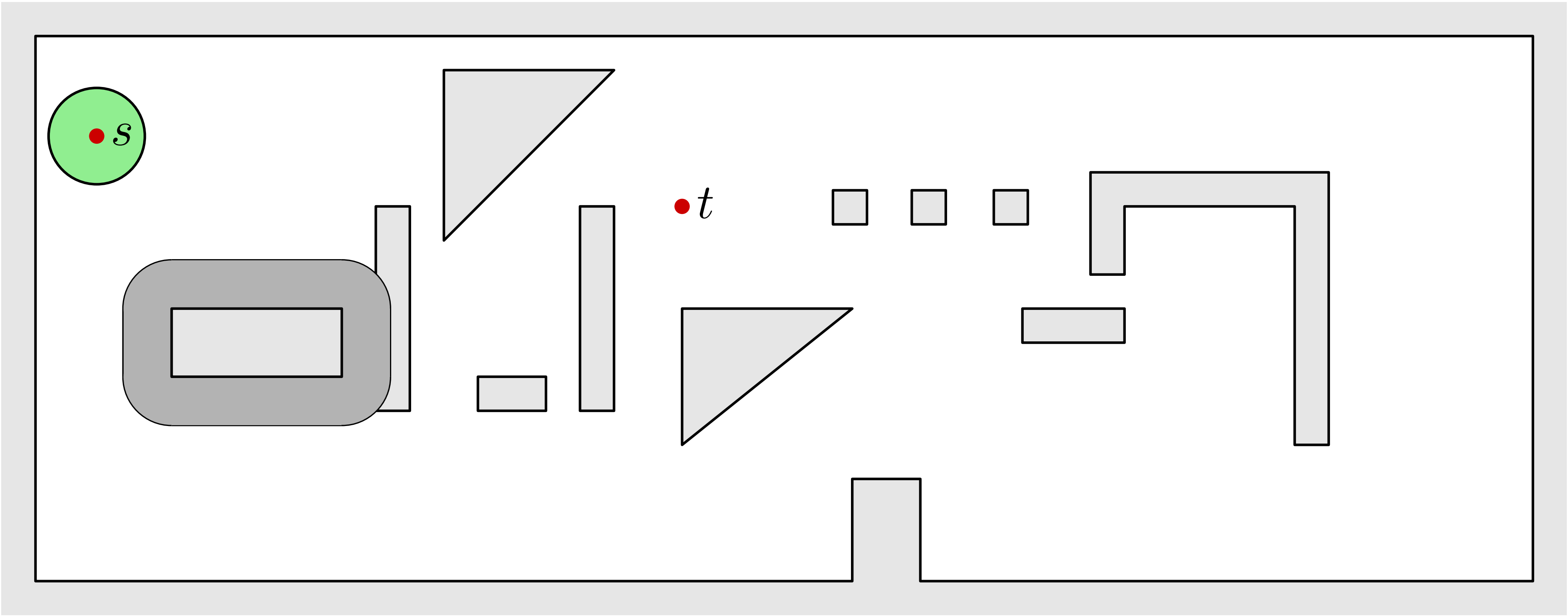


Robot Motion Planning





1. Transform problem to motion-planning problem for a point-shaped robot



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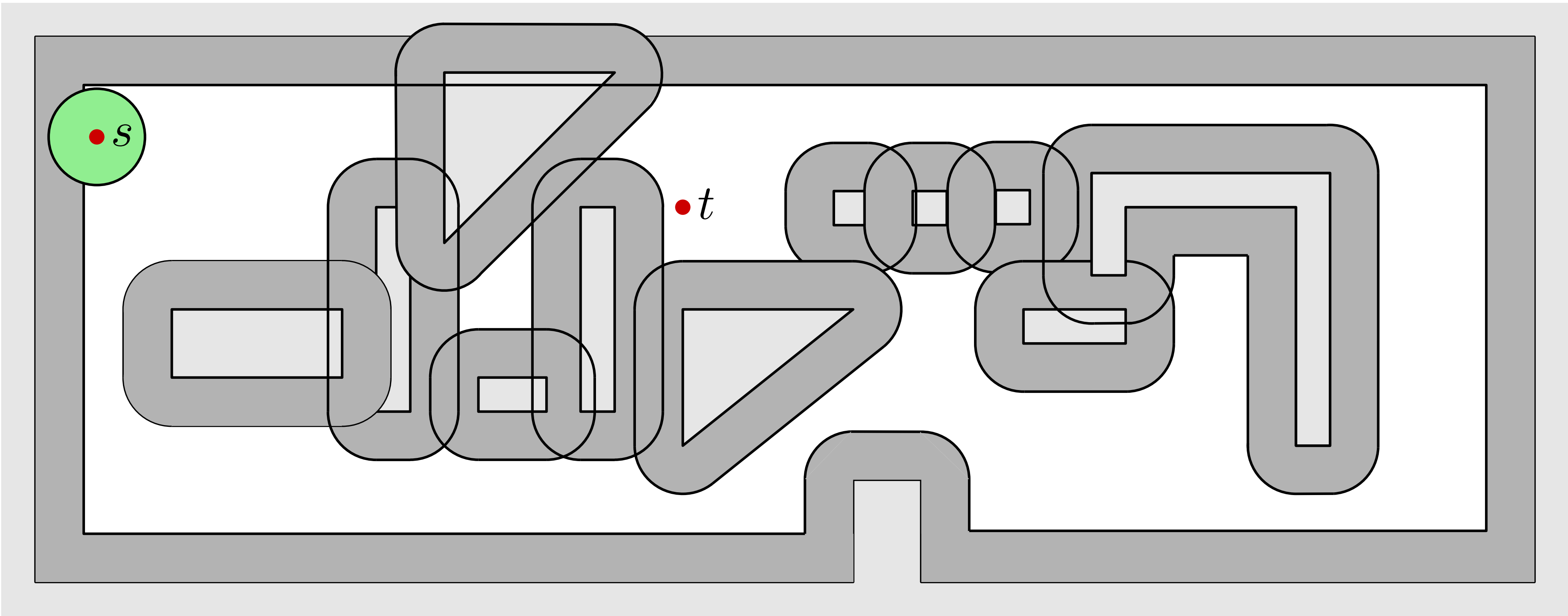

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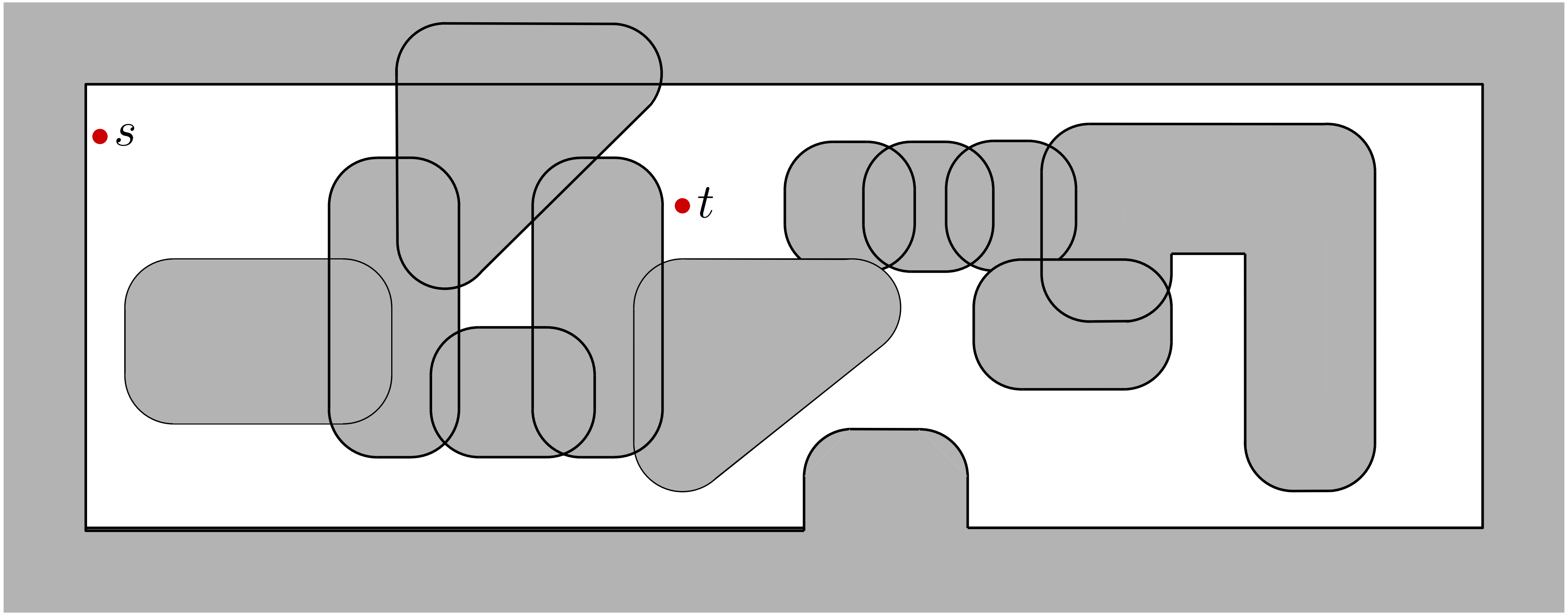
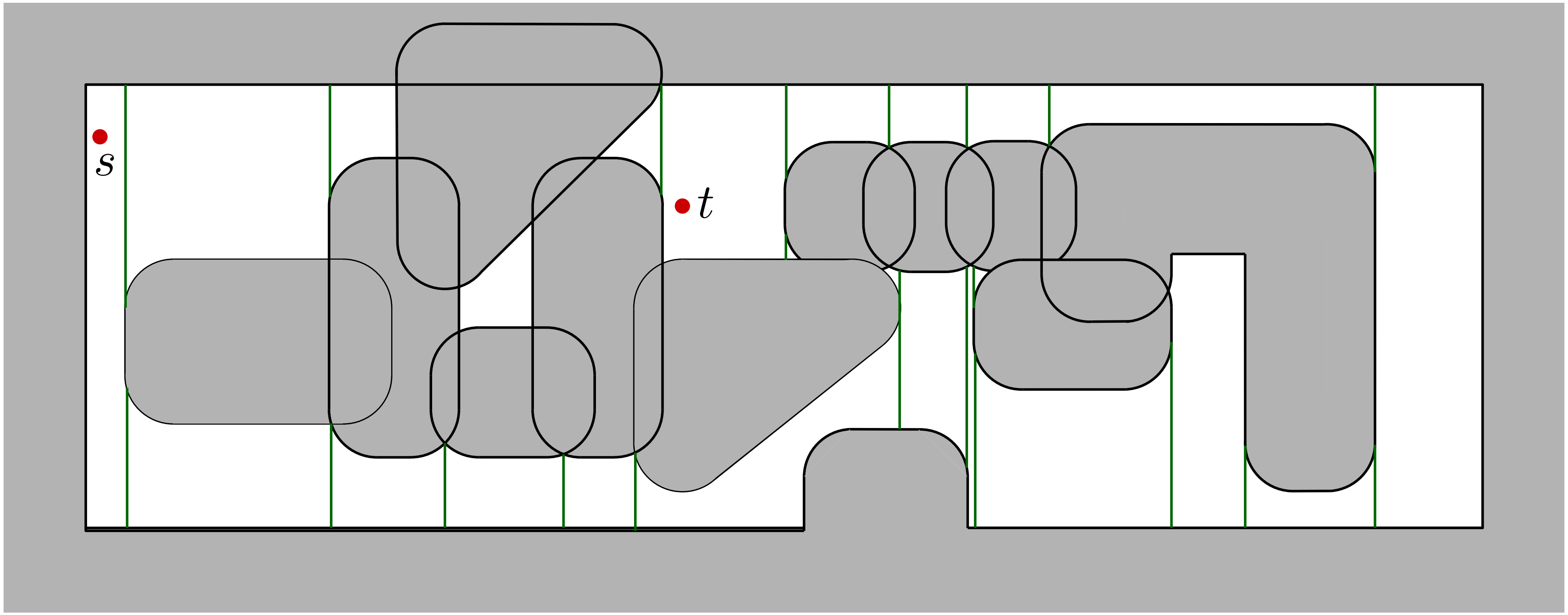
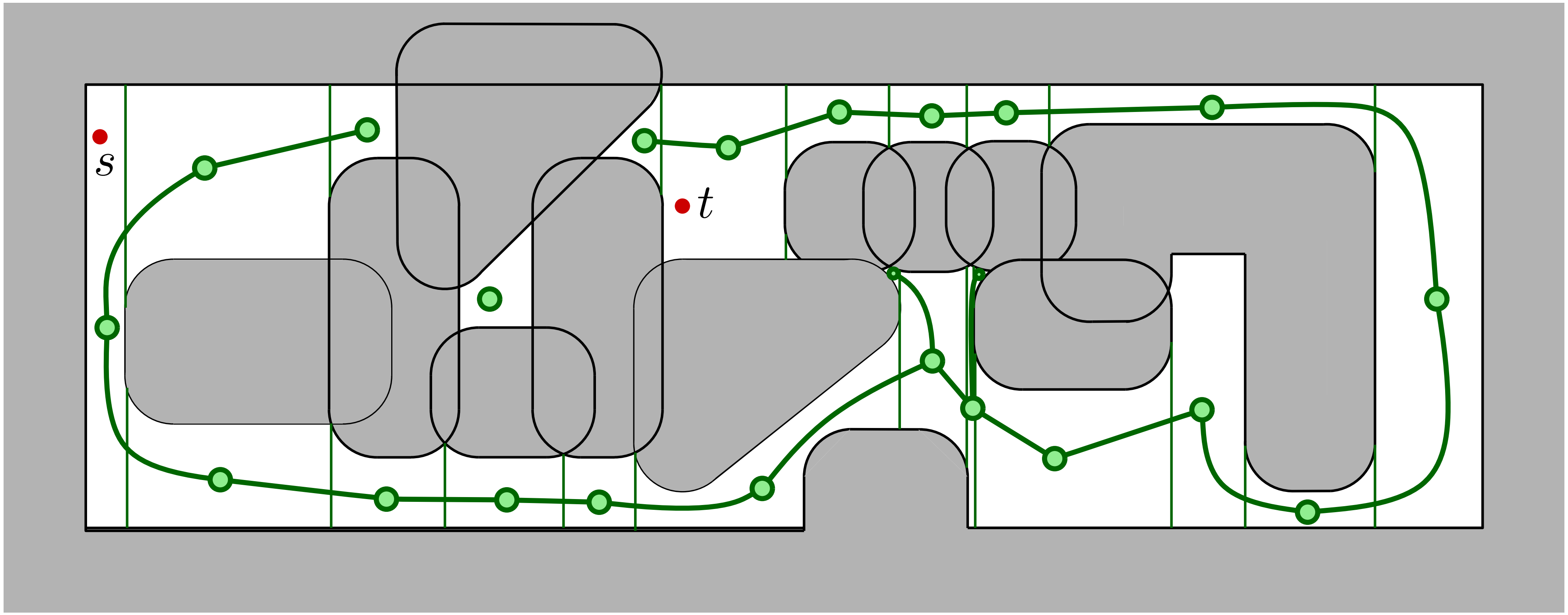
Robot Motion Planning



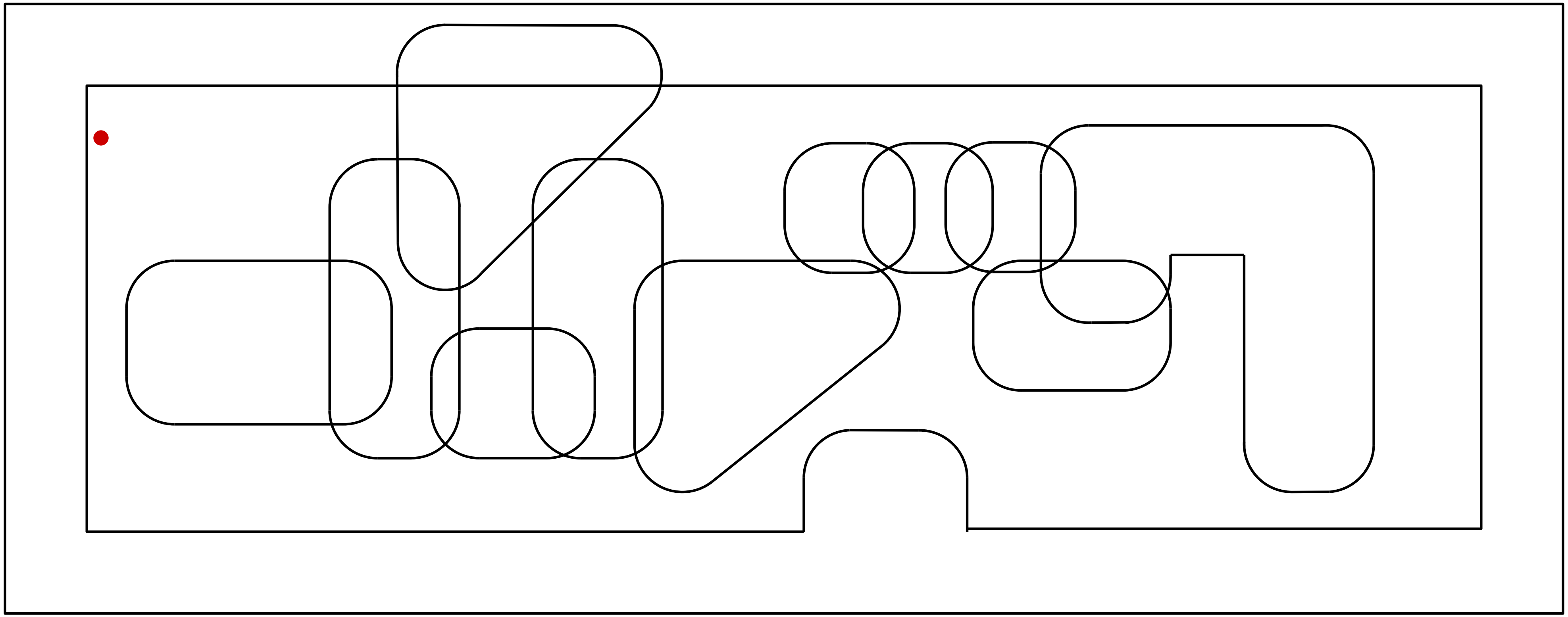
1. Transform problem to motion-planning problem for a point-shaped robot by expanding each obstacle. (Expanded obstacles can intersect!)



1. Transform problem to motion-planning problem for a point-shaped robot by expanding each obstacle. (Expanded obstacles can intersect!)
2. Decompose free space into "quadrilaterals"



1. Transform problem to motion-planning problem for a point-shaped robot by expanding each obstacle. (Expanded obstacles can intersect!)
2. Decompose free space into "quadrilaterals"
3. Construct motion graph $\mathcal{G}$ and compute path from s to t in $\mathcal{G}$



reachable region of the robot

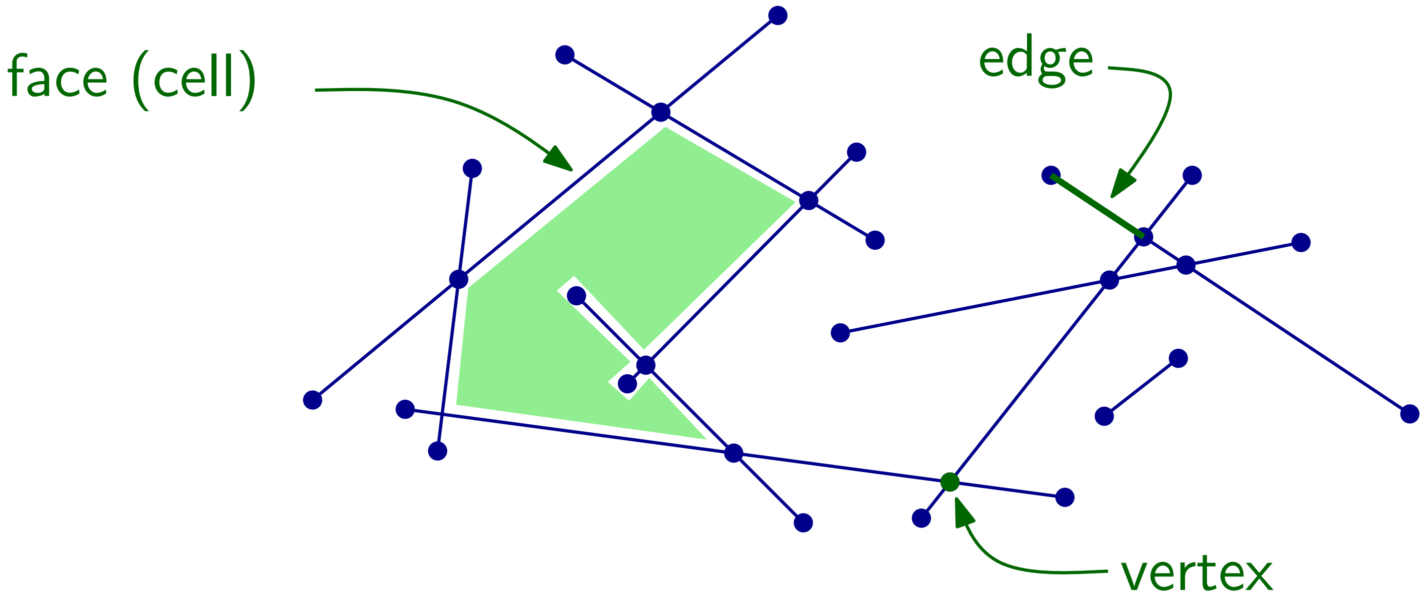
$=$

single cell in arrangement induced by a set S of n curves in $\mathbb{R}^2$

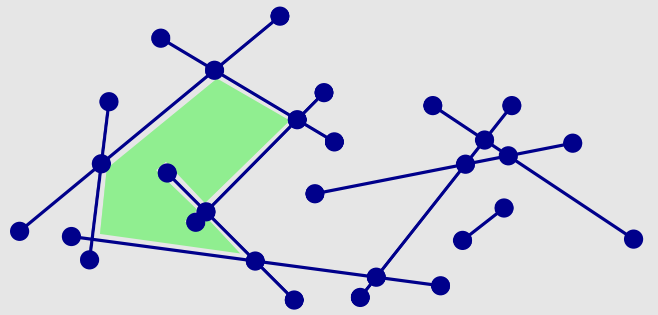
for other types of robots: in $\mathbb{R}^d$, where $d = \#$(degrees of freedom)

S: set of n lines / segments / curves / etc in $\mathbb{R}^2$

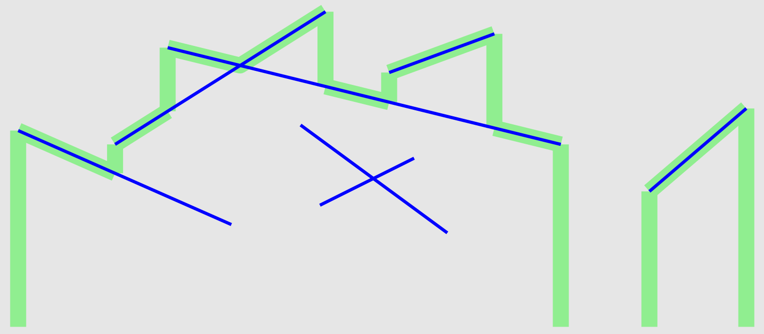
$\mathcal{A}(S)$ = arrangement induced by S
= partitioning of $\mathbb{R}^2$ into faces, edges, and vertices induced by S



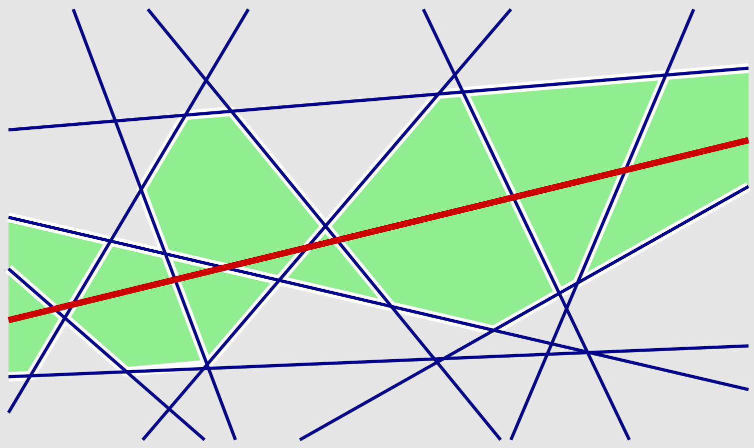
combinatorial complexity of $\mathcal{A}(S)$ = total number of vertices, edges, faces



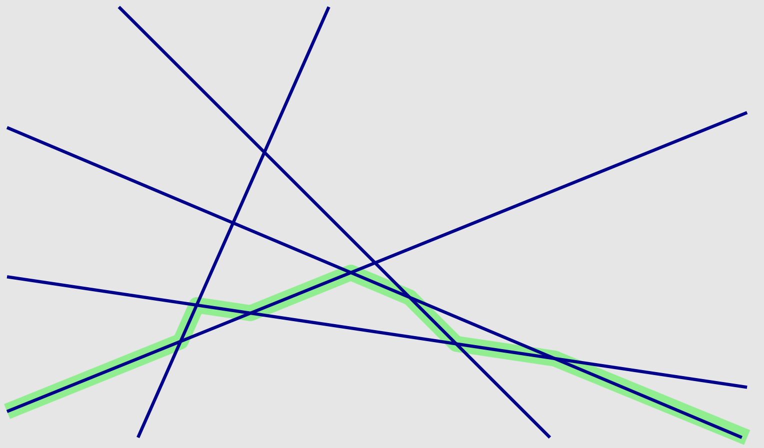
single cell



upper envelope



zone



k-level

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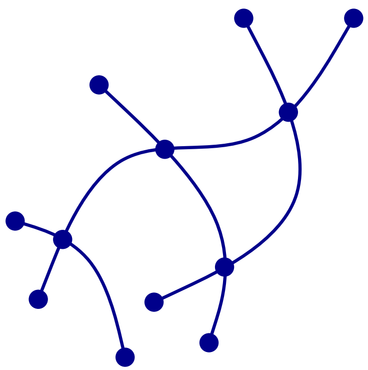

Assume curves are finite.

- number of vertices
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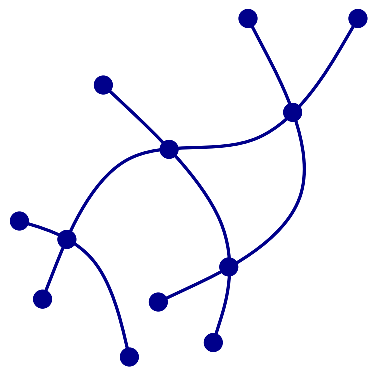

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- number of vertices $|V| \leqslant 2n + s \cdot \binom{n}{2} = O(n^2)$
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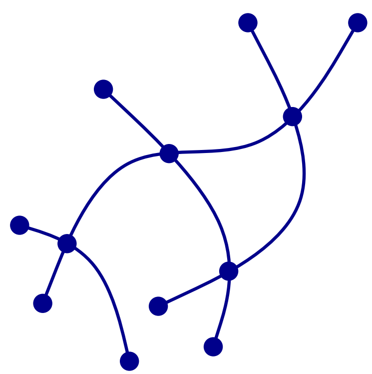

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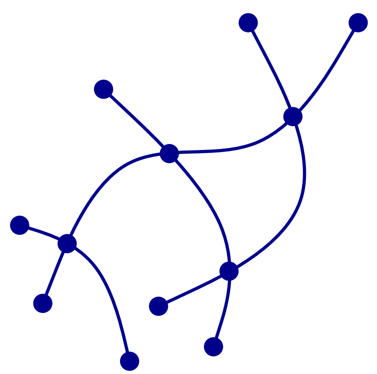

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Euler's formula:

$|V| - |E| + |F| = 2$

■

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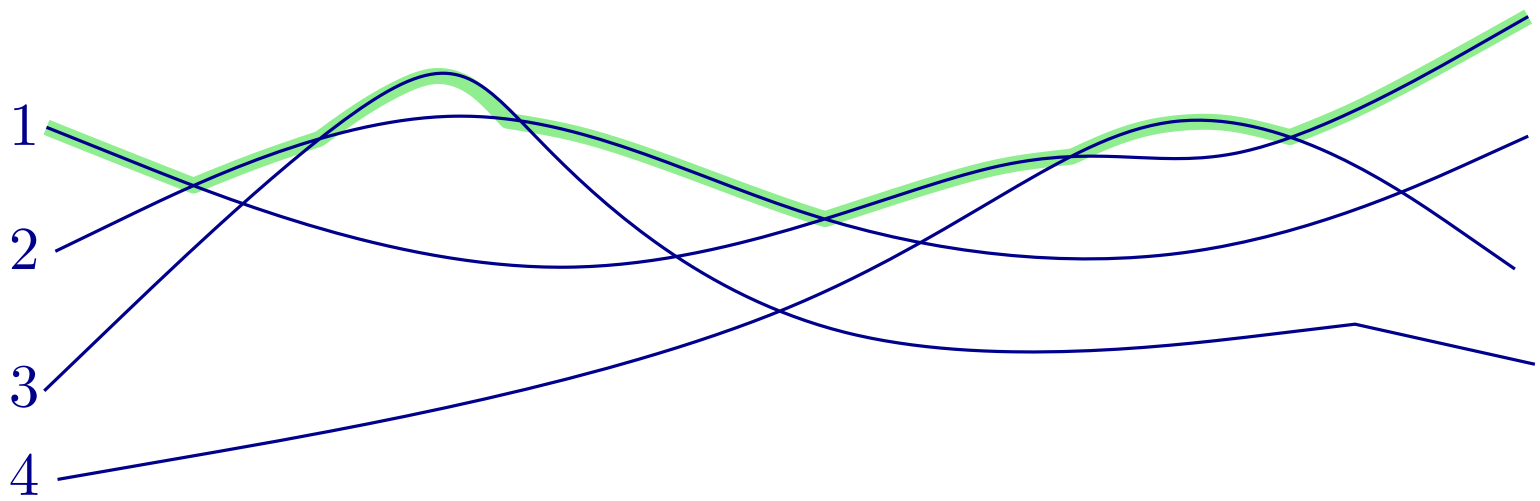

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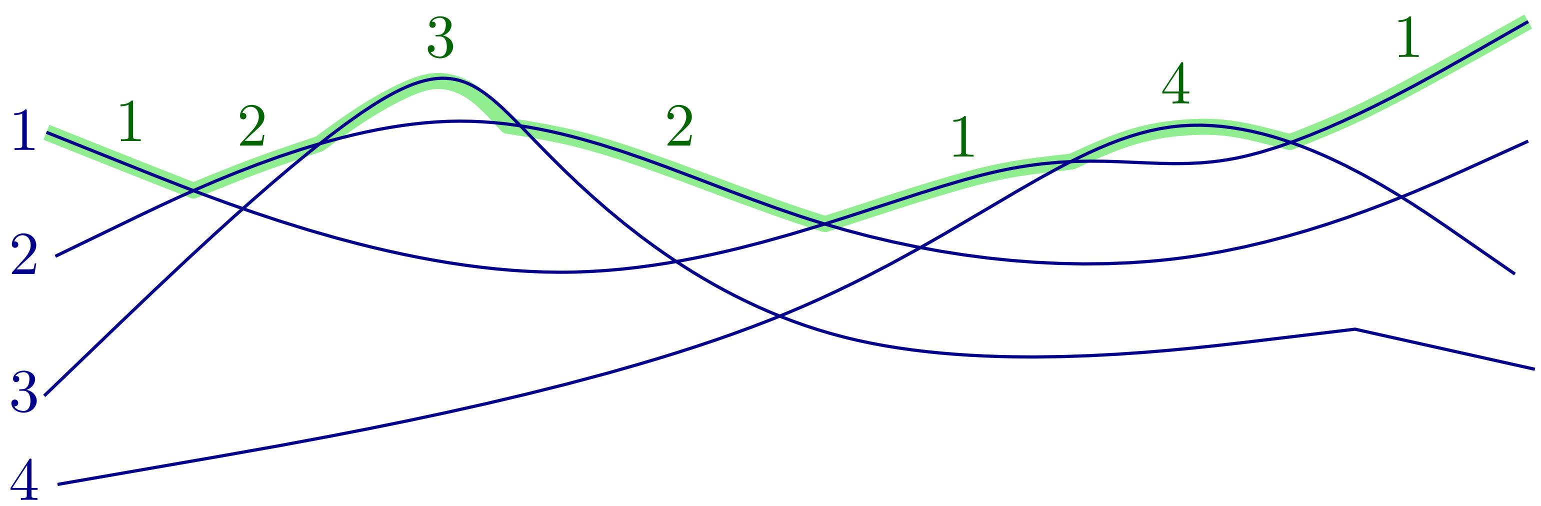

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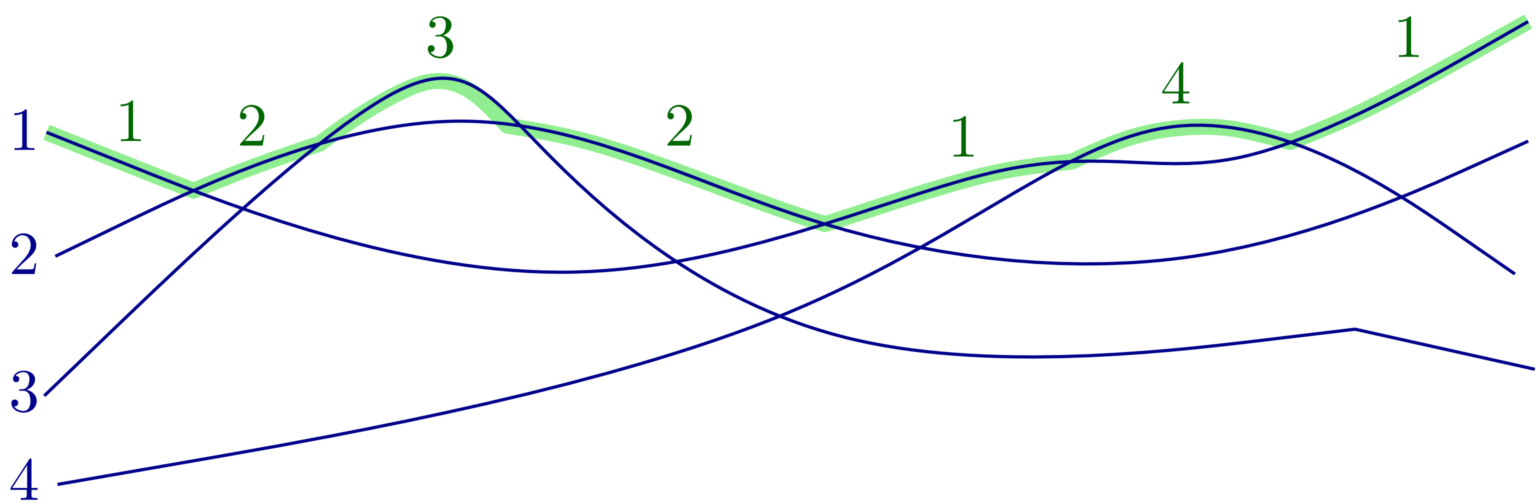
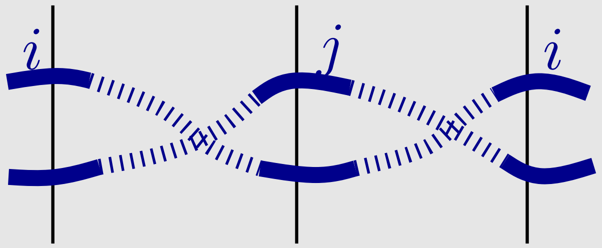


alternating sequence of length t implies $t-1$ intersections



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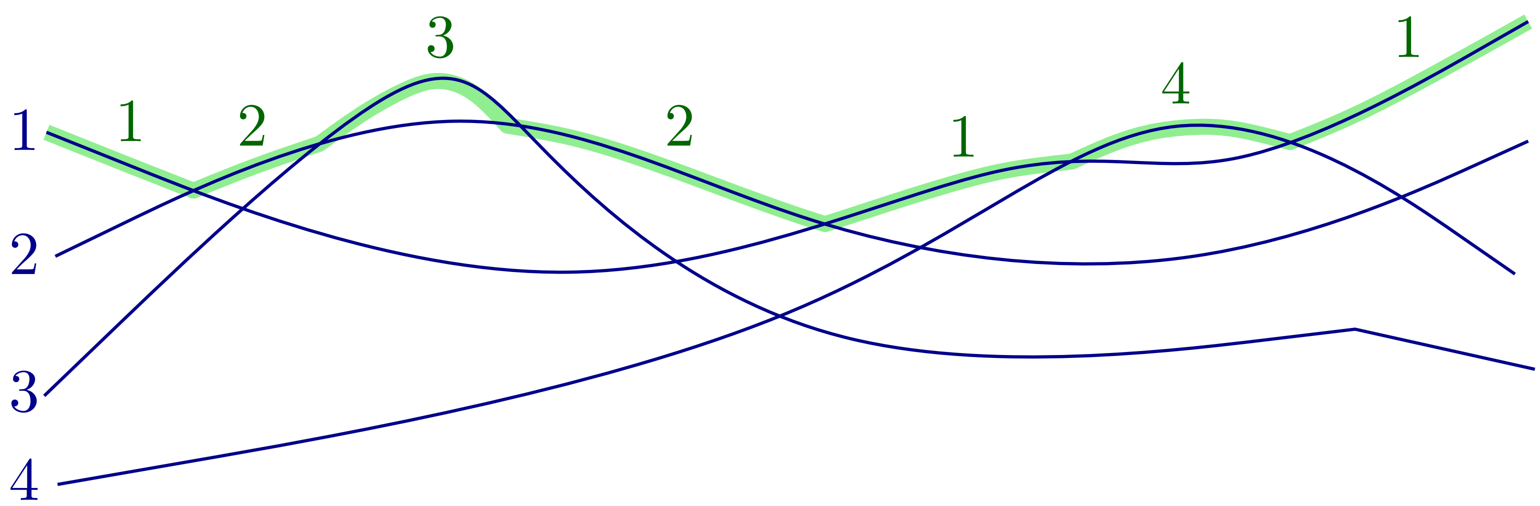
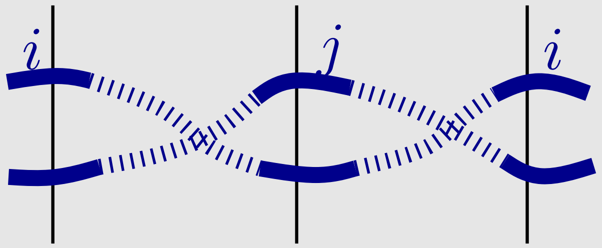


we cannot have alternating sequence of length $s+2$
$\Longrightarrow$ DS(n, s)-sequence

alternating sequence of length t implies $t-1$ intersections



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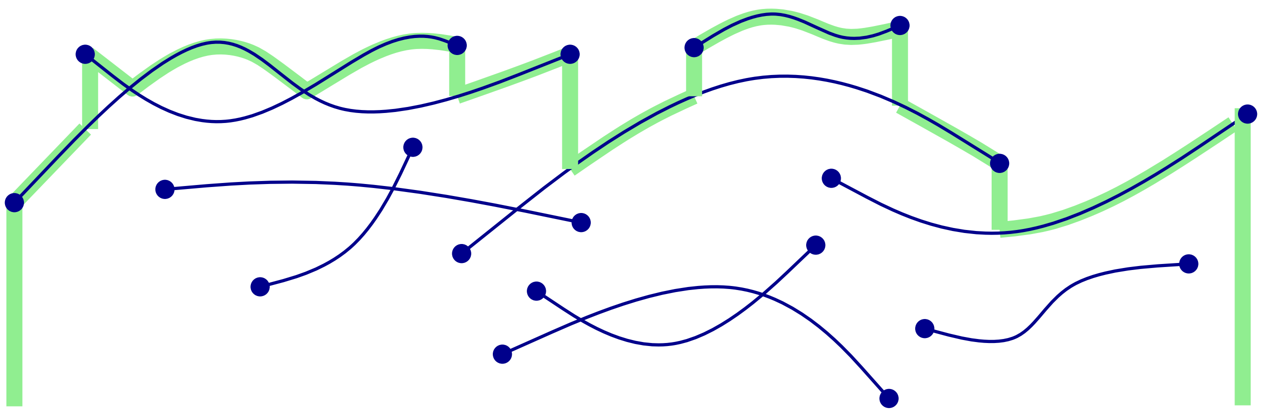

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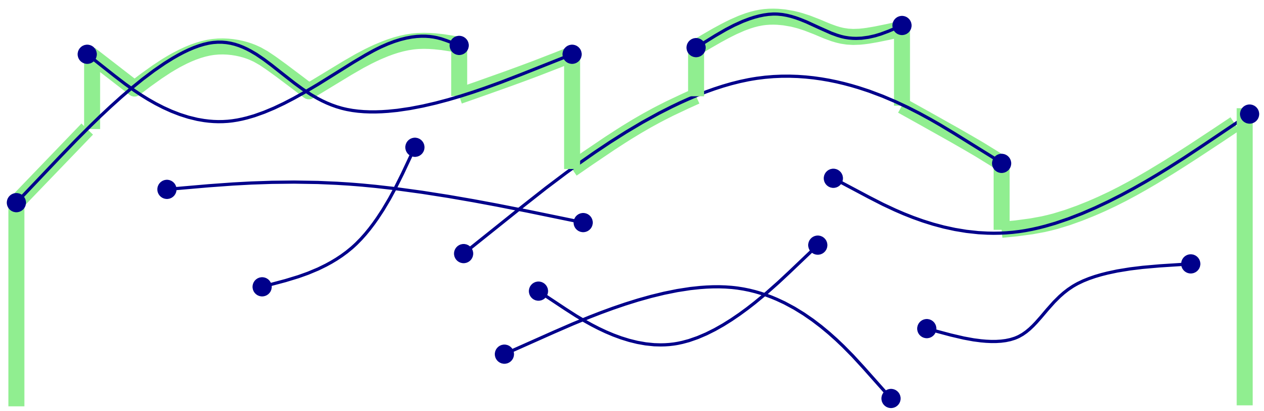
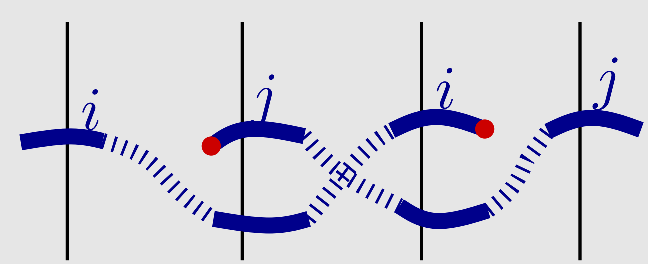


alternating sequence of length t implies $t-3$ intersections



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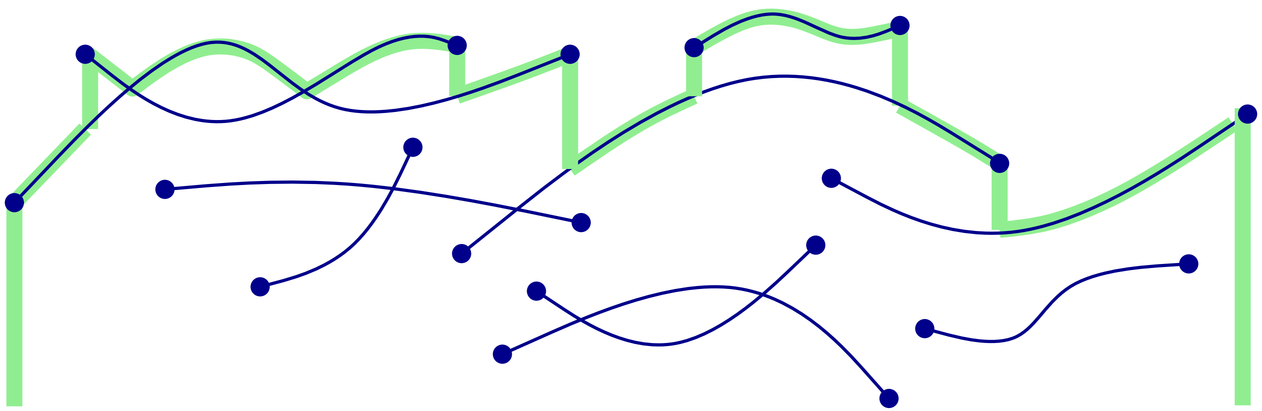

we cannot have alternating sequence of length $s+4$
$\Longrightarrow$ DS(n, $s+2$)-sequence

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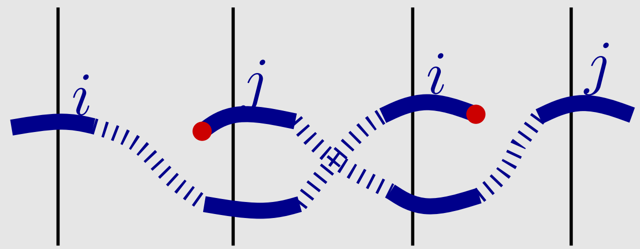
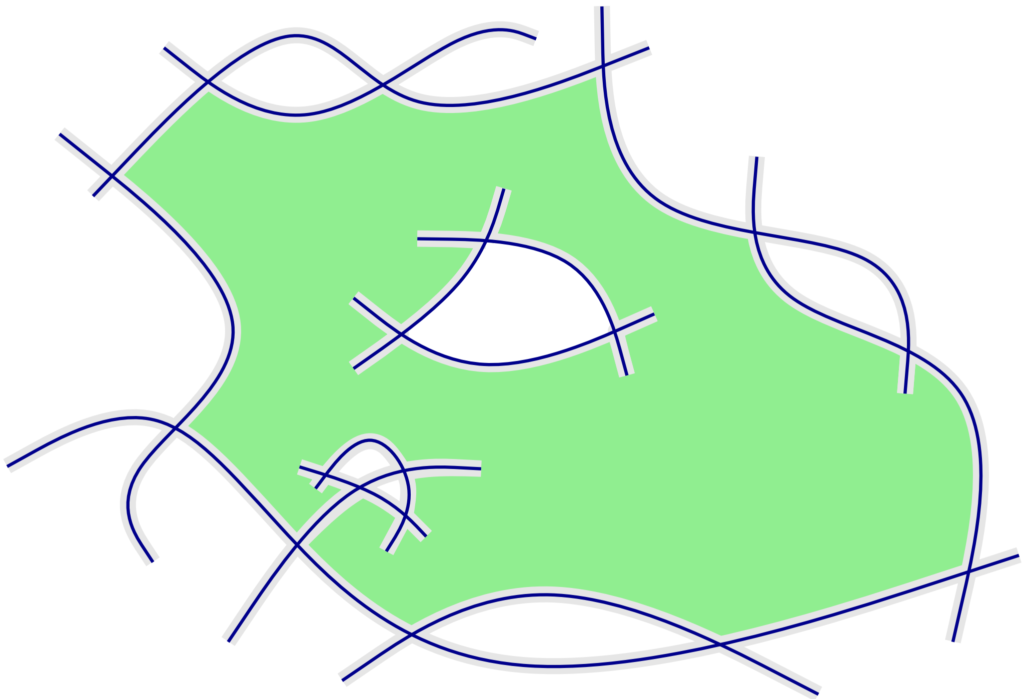
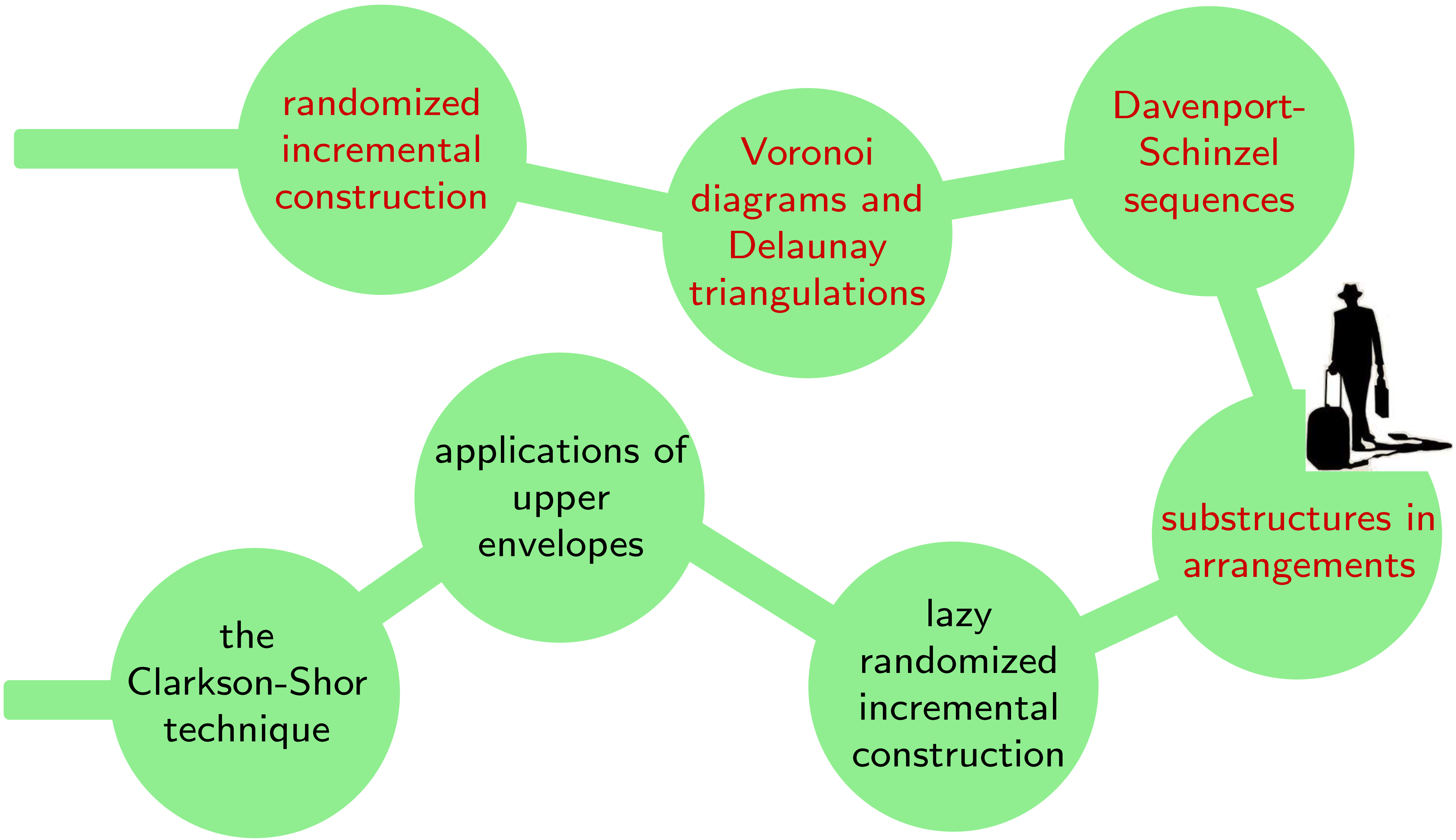


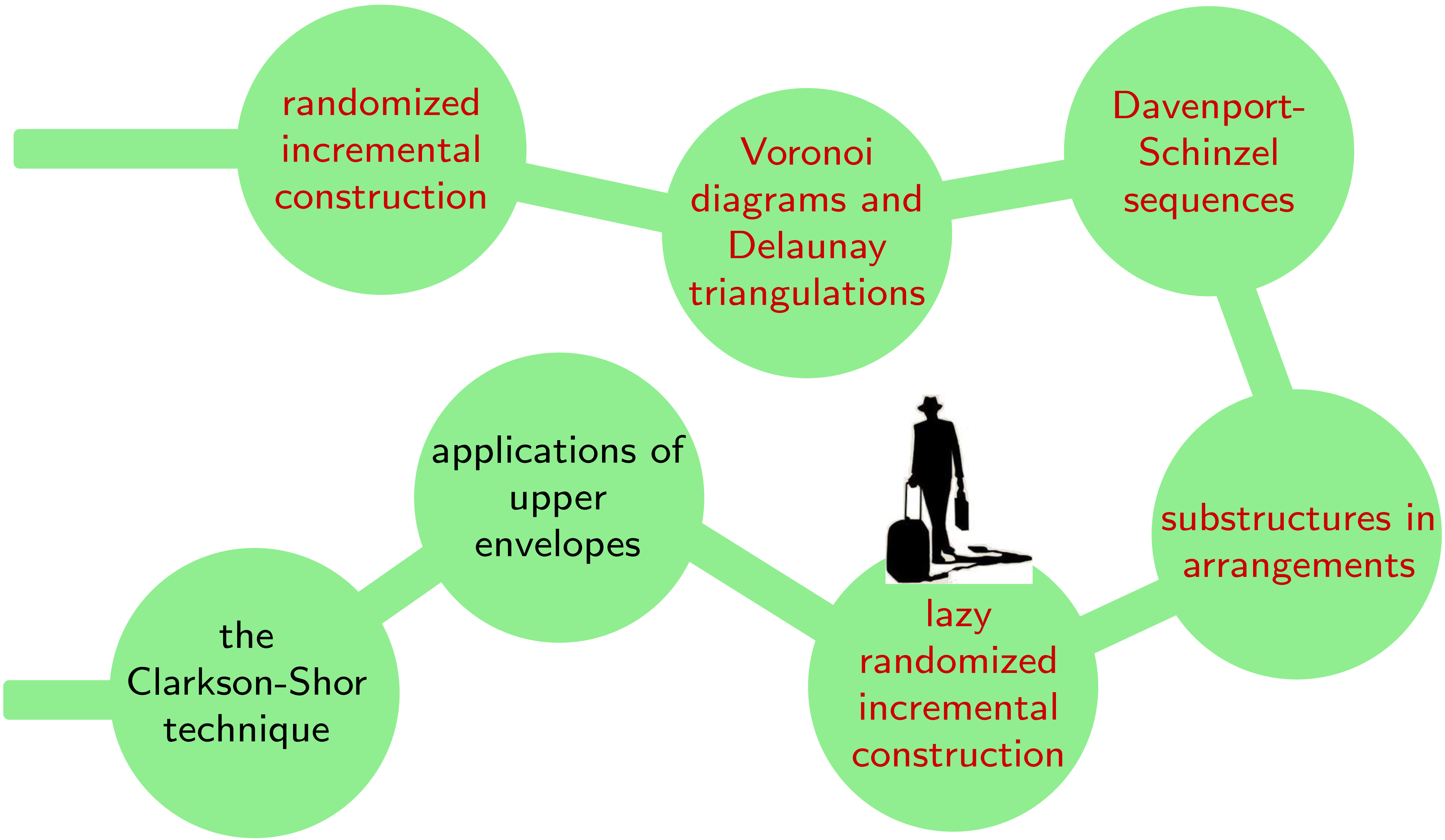
■

Theorem. Let S be a set of n curves in the plane such that any two curves intersect at most s times. Then the maximum complexity of a single cell of $\mathcal{A}(S)$ is $O(\mathrm{DS}_{s+2}(n))$.




randomized incremental construction
Voronoi diagrams and Delaunay triangulations
Davenport-Schinzel sequences
substructures in arrangements
lazy randomized incremental construction
applications of upper envelopes
the Clarkson-Shor technique

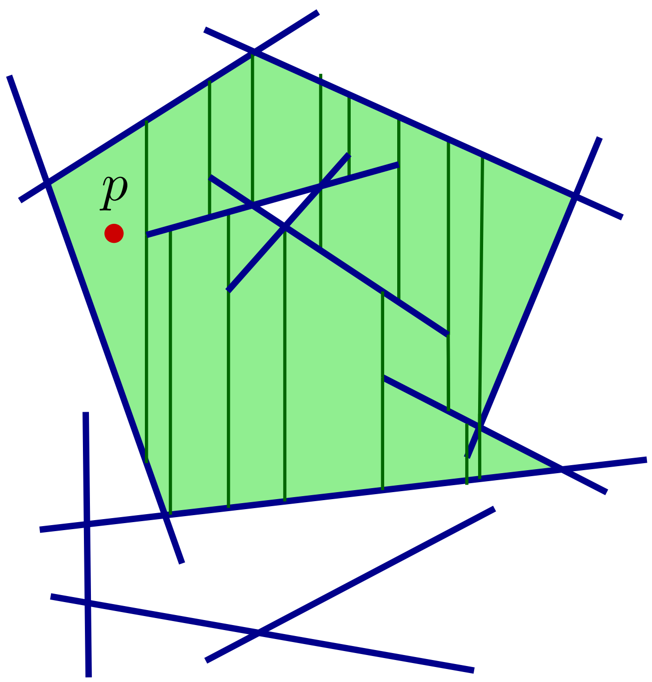
Course Overview



Computing a single cell with RIC?

Input: Set S of n segments in the plane, and a point p

Goal: Compute the face of $\mathcal{A}(S)$ containing p



The RIC framework

- S = set of n input objects
- $\mathcal{C}(S)$ = set of configurations defined by S
  - $D(\Delta) \subset S$ = defining set of $\Delta \in \mathcal{C}(S)$
    size bounded by fixed constant
  - $K(\Delta) \subset S$ = conflict list of $\Delta \in \mathcal{C}(S)$
- Goal: Compute $\mathcal{C}_{\text{act}}(S) =$
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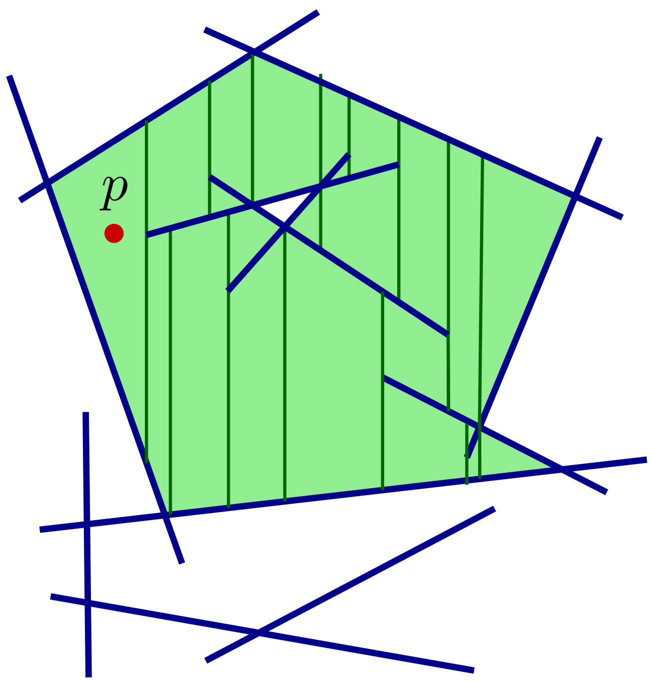

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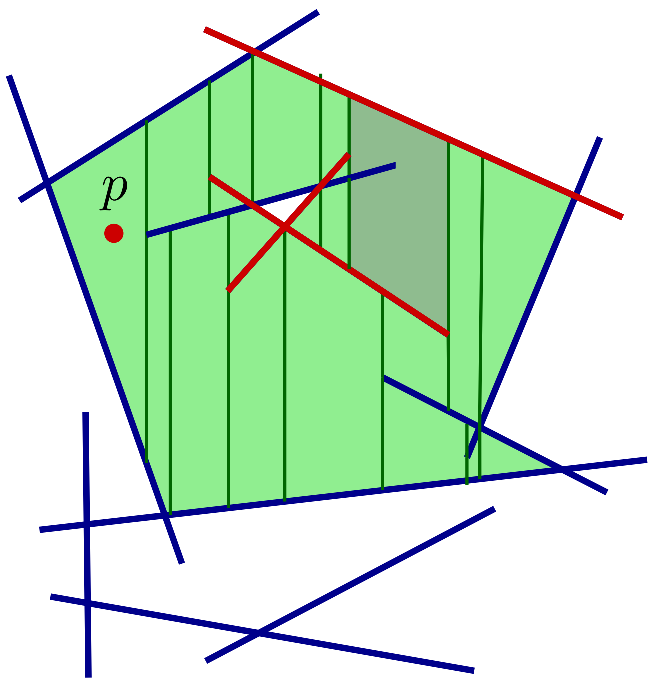

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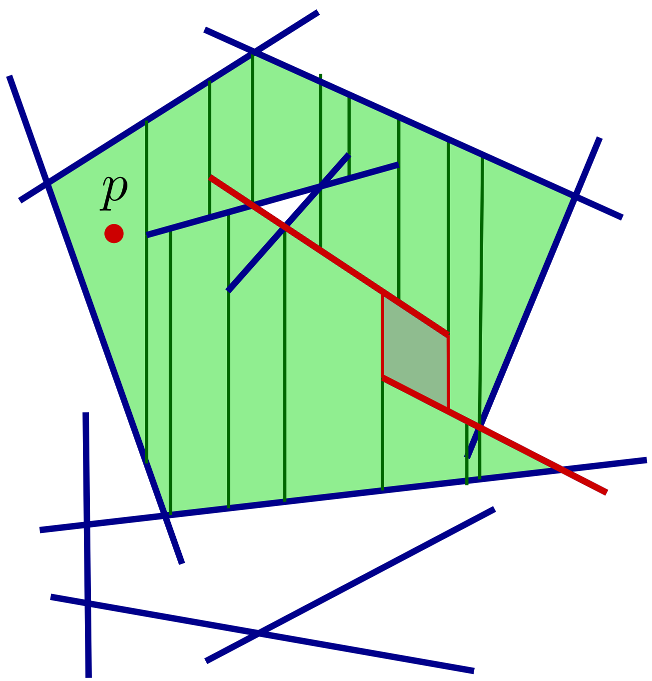

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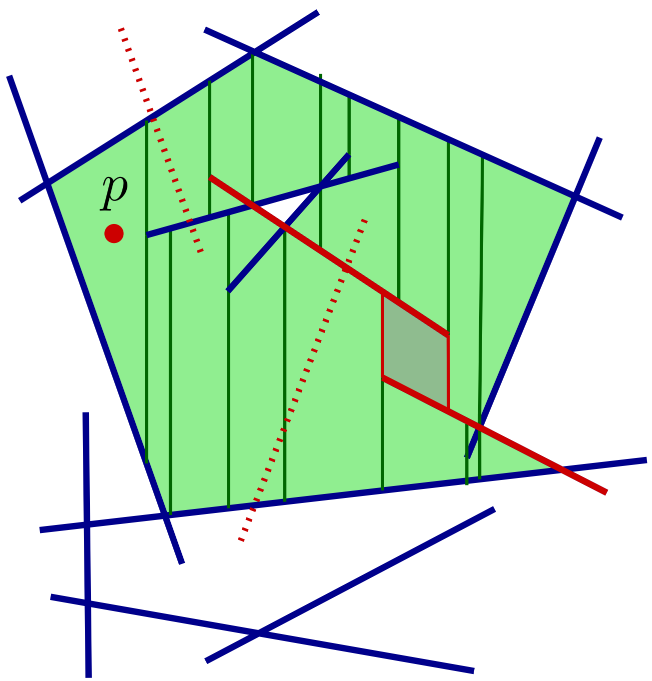

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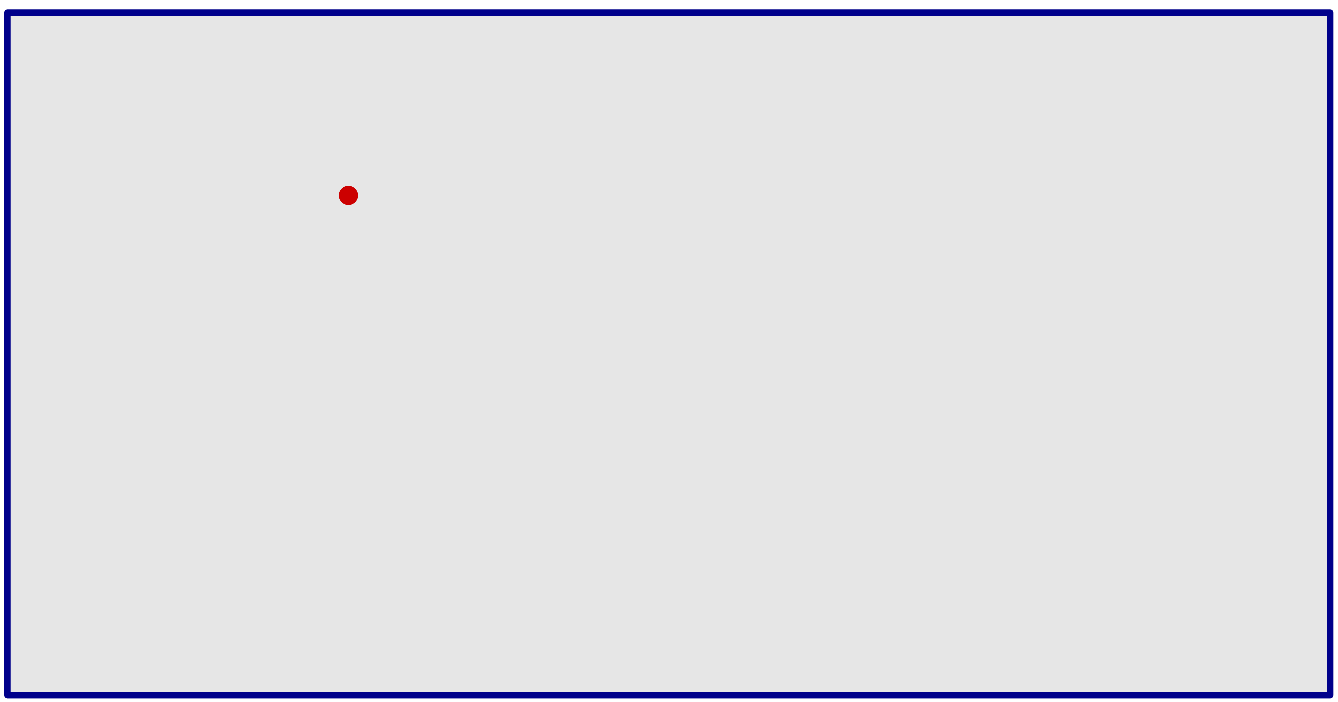

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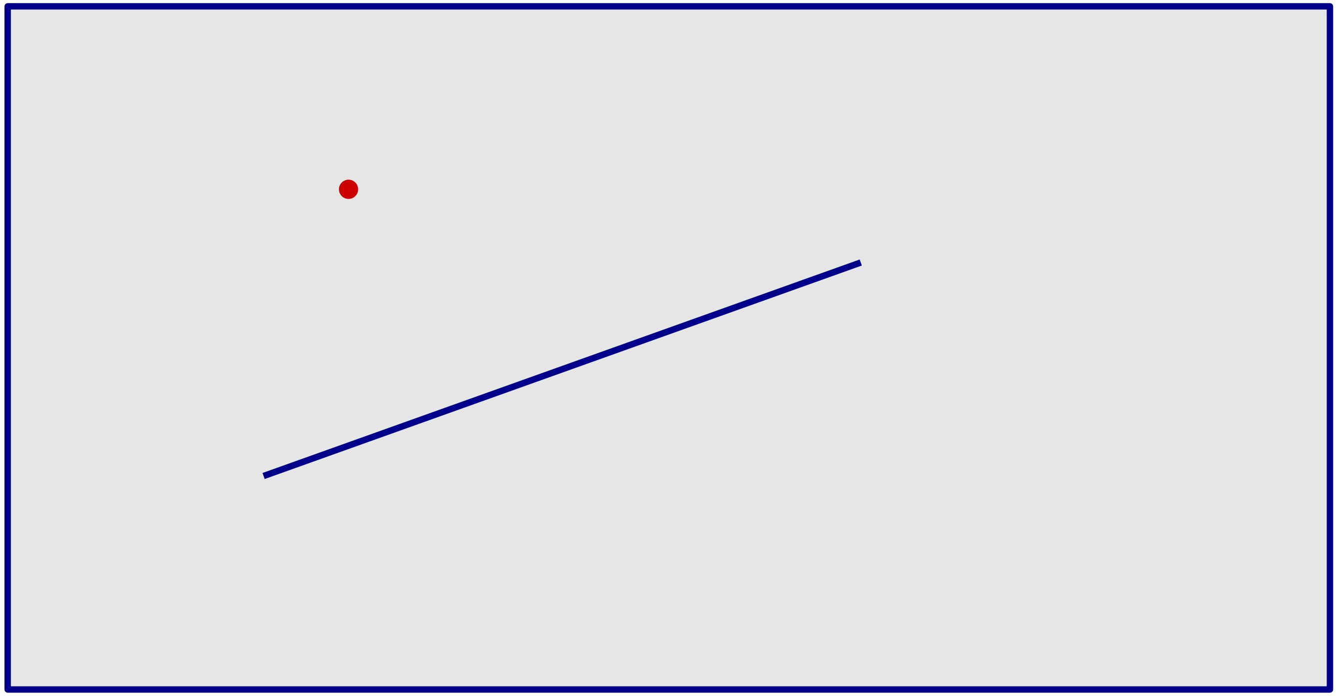

Lazy Randomized Incremental Construction

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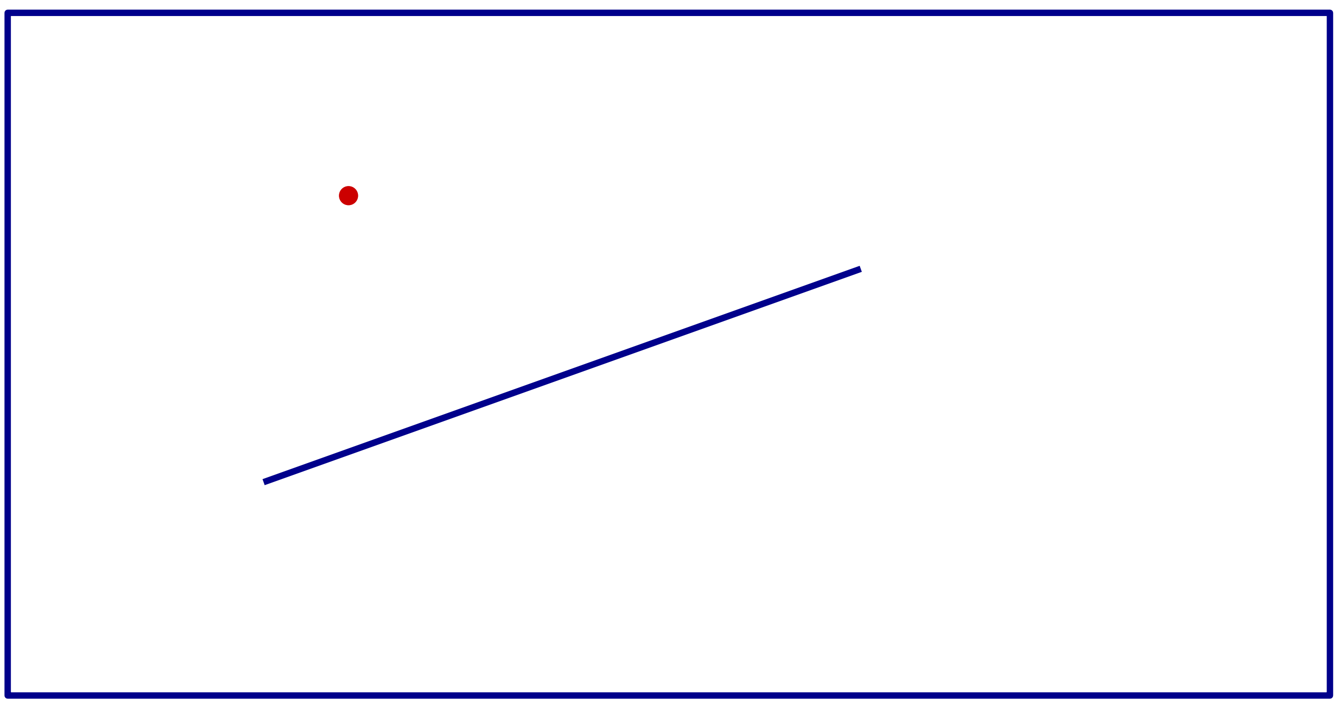

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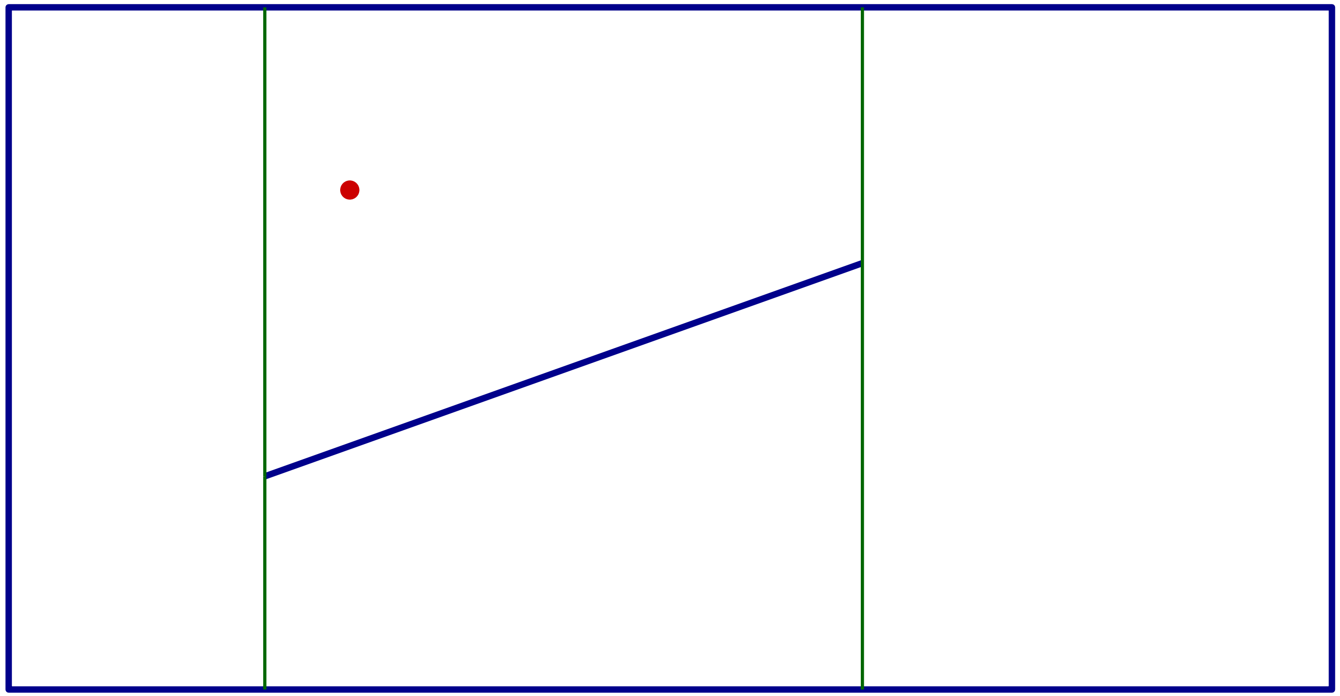

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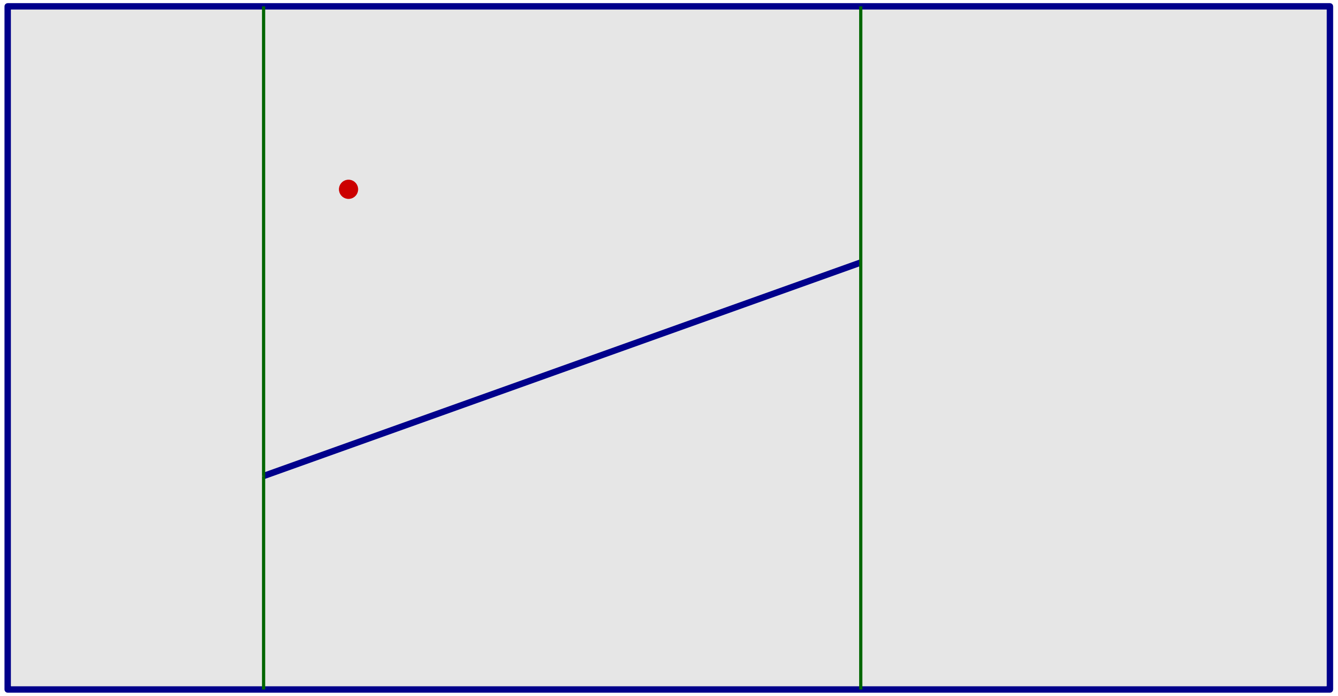

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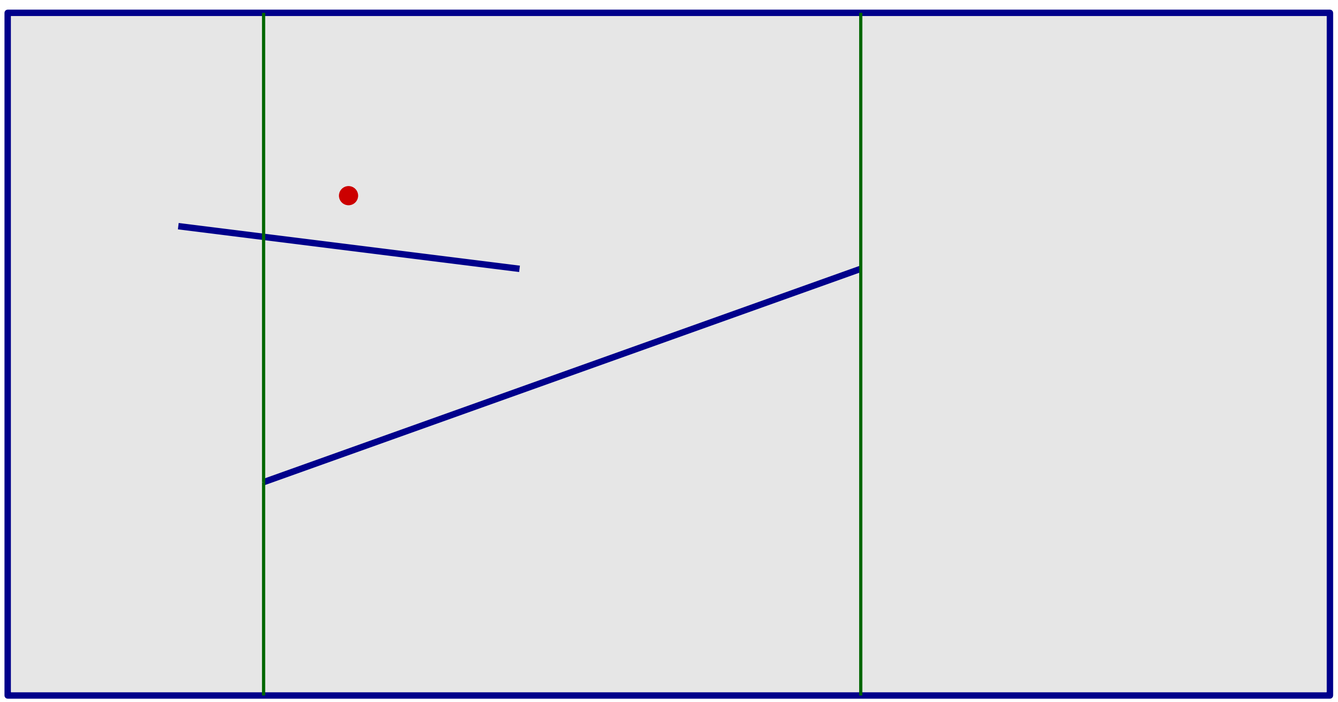

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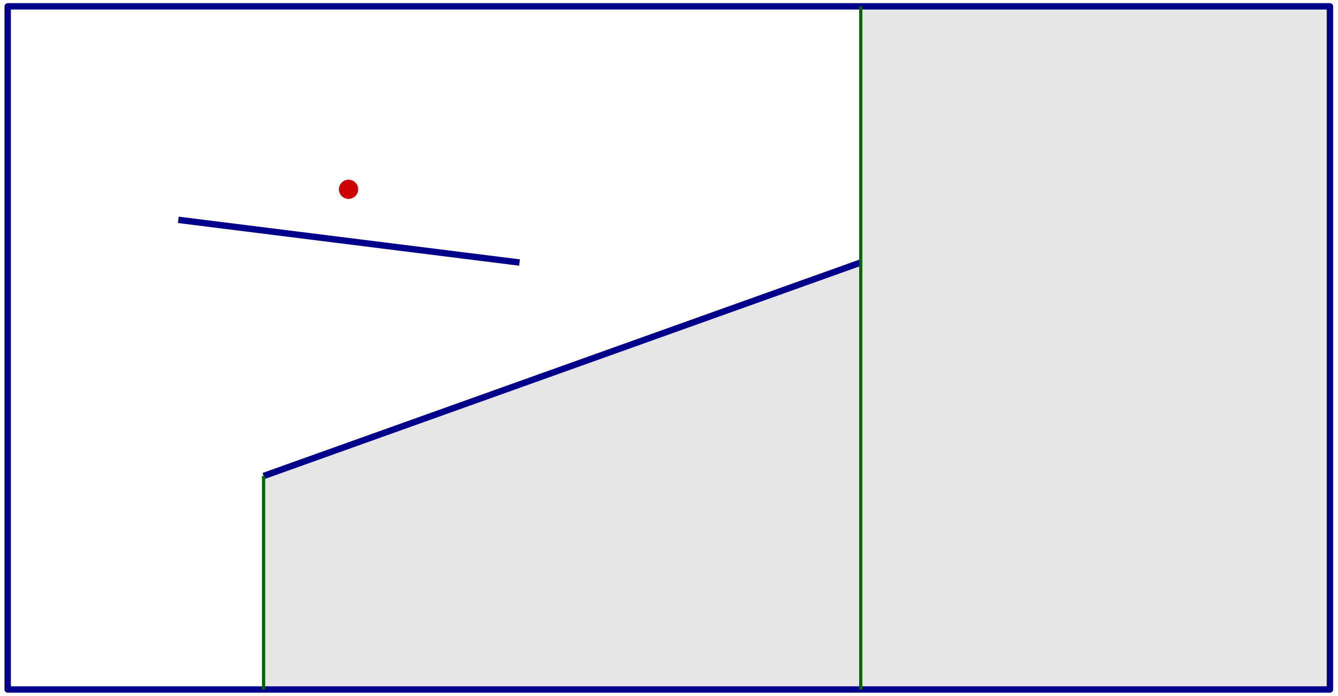

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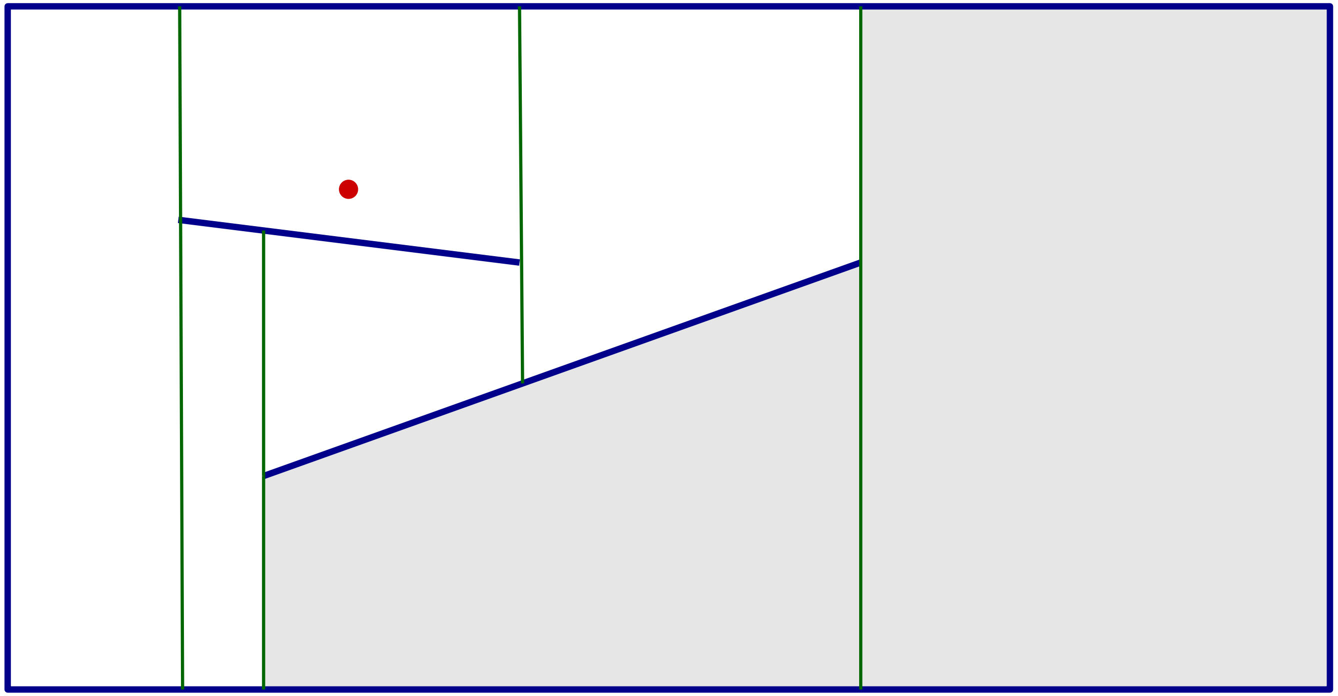

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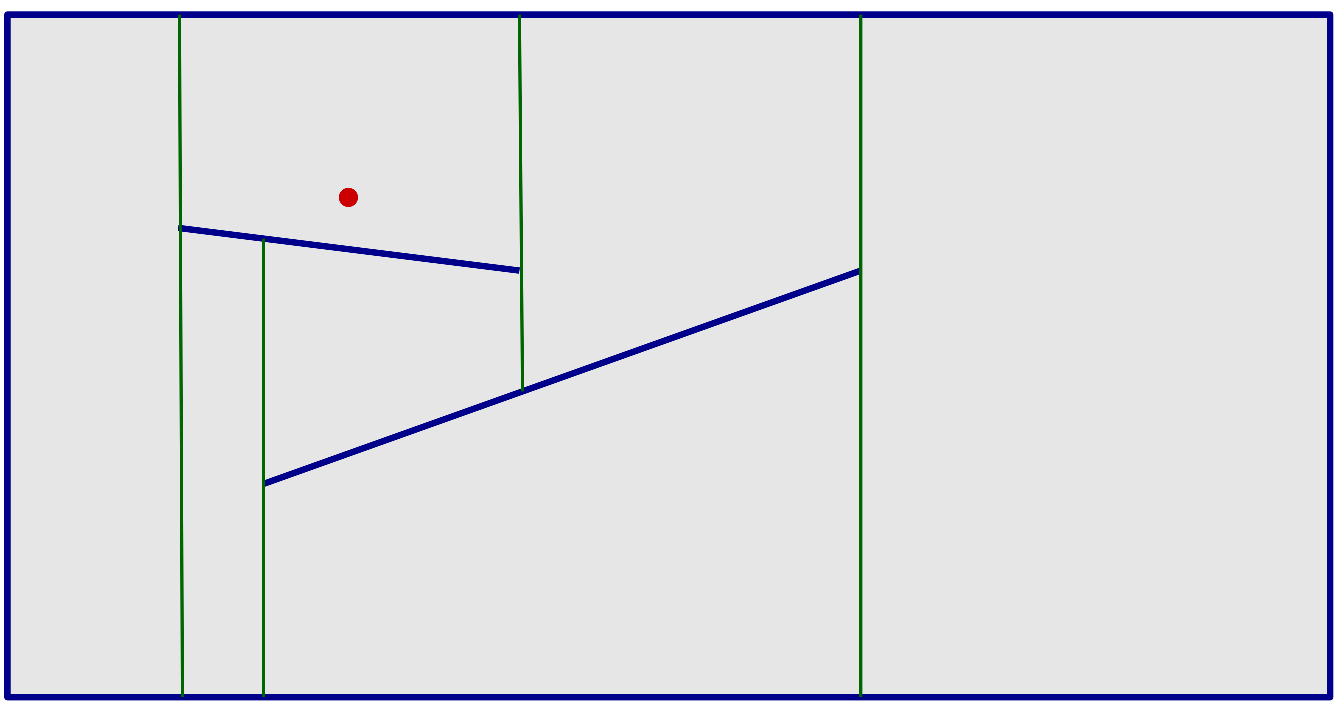

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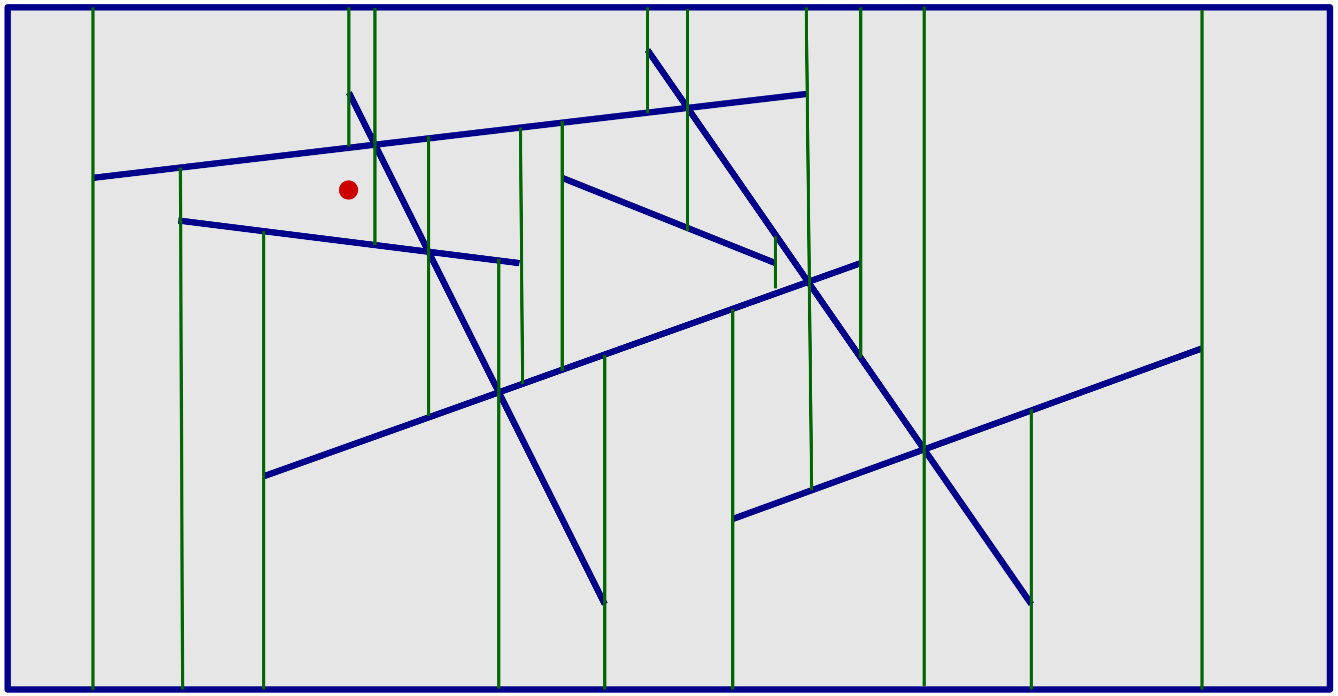

after 7 iterations

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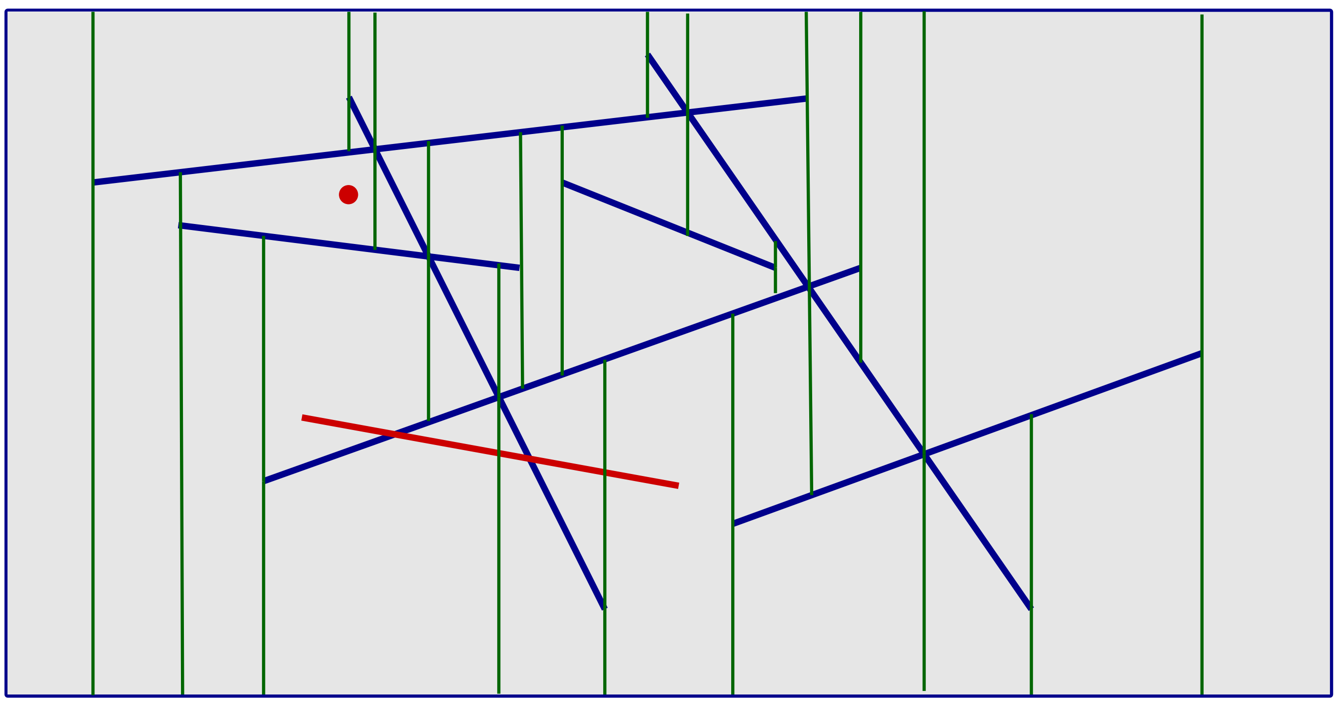

the 8-th iteration

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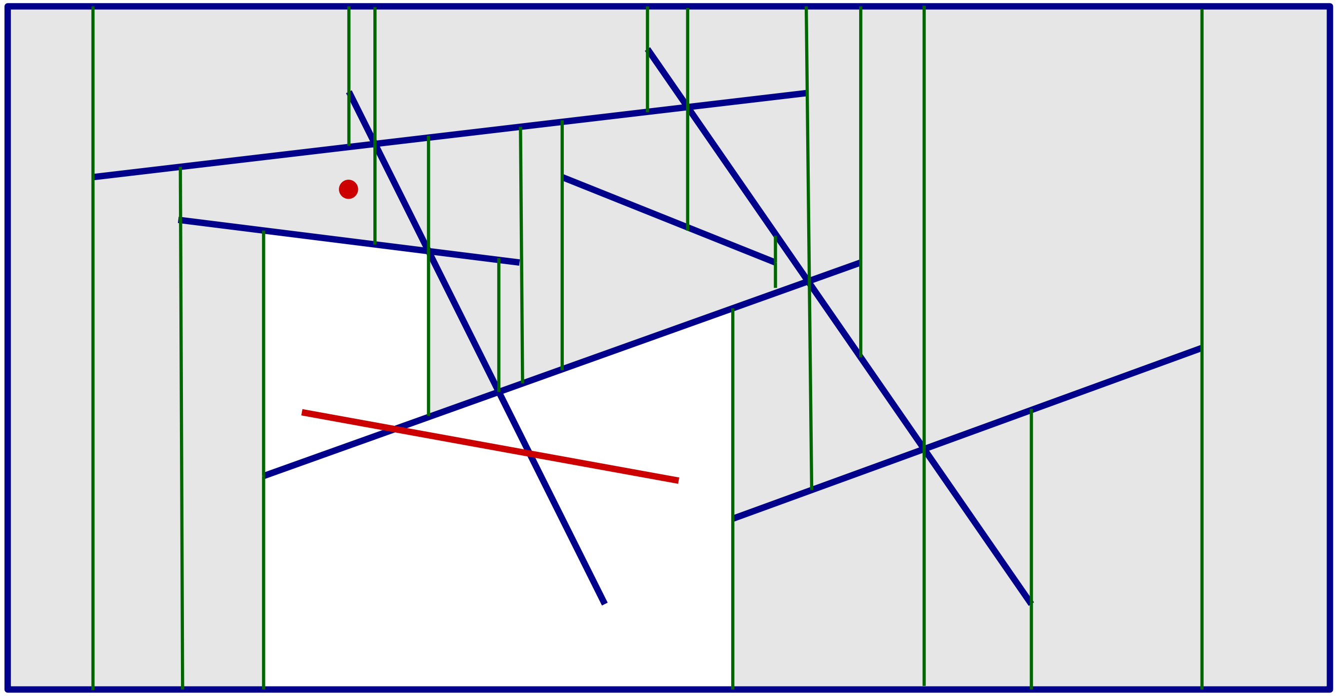

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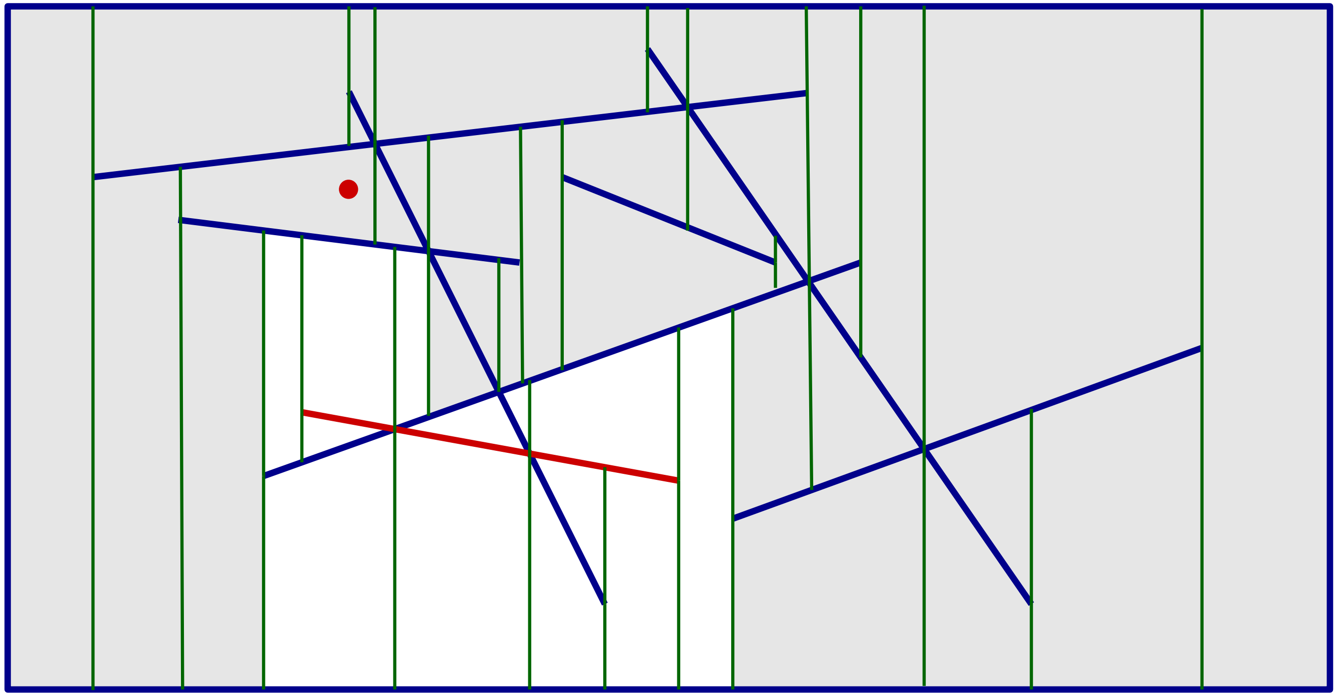

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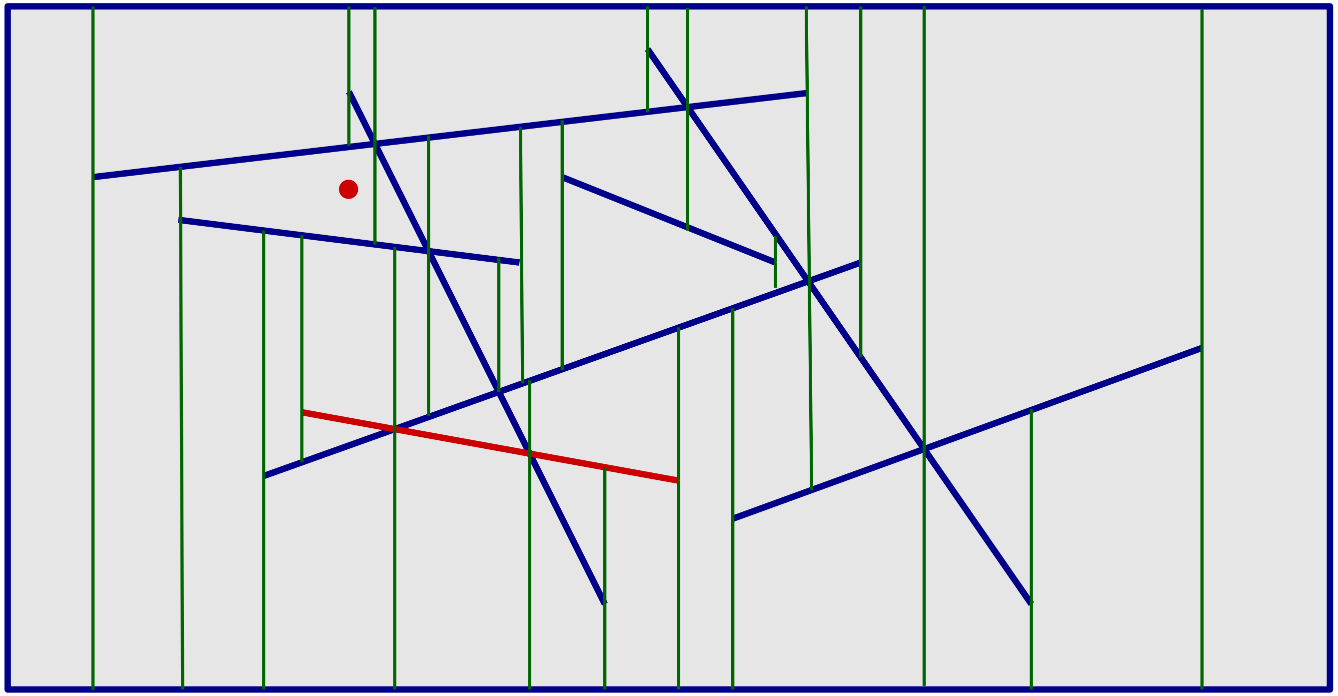

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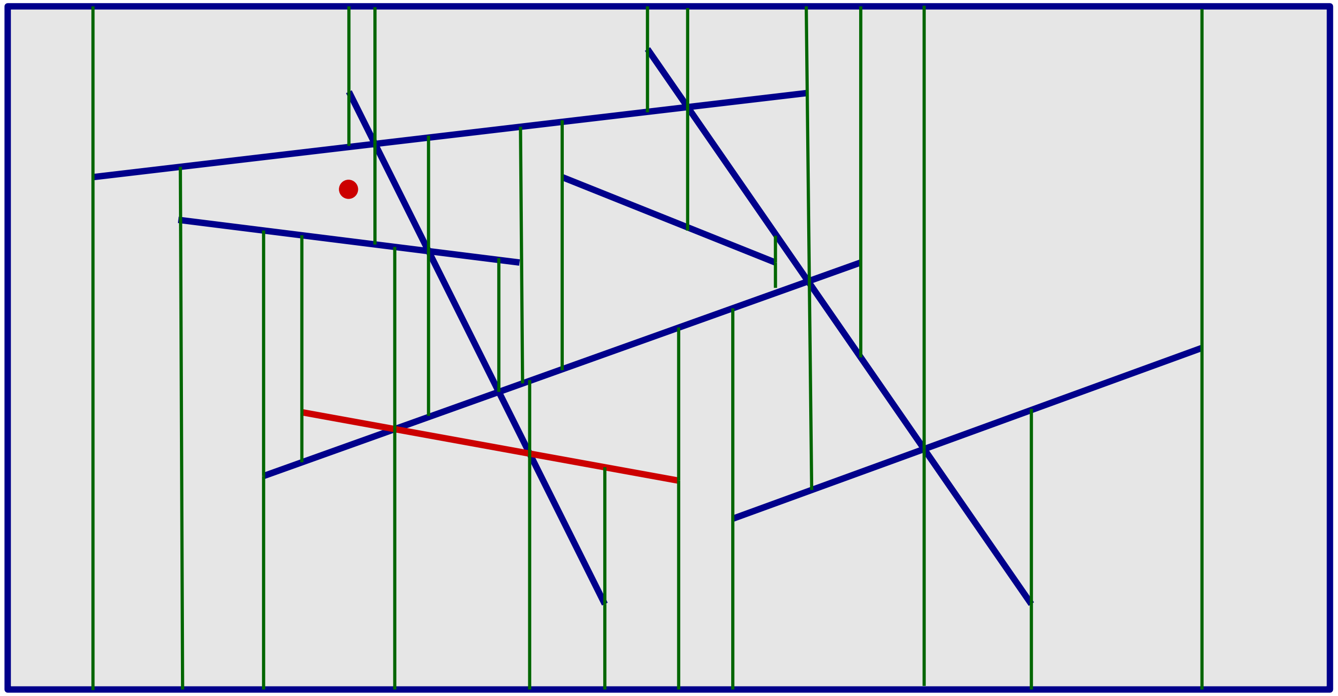
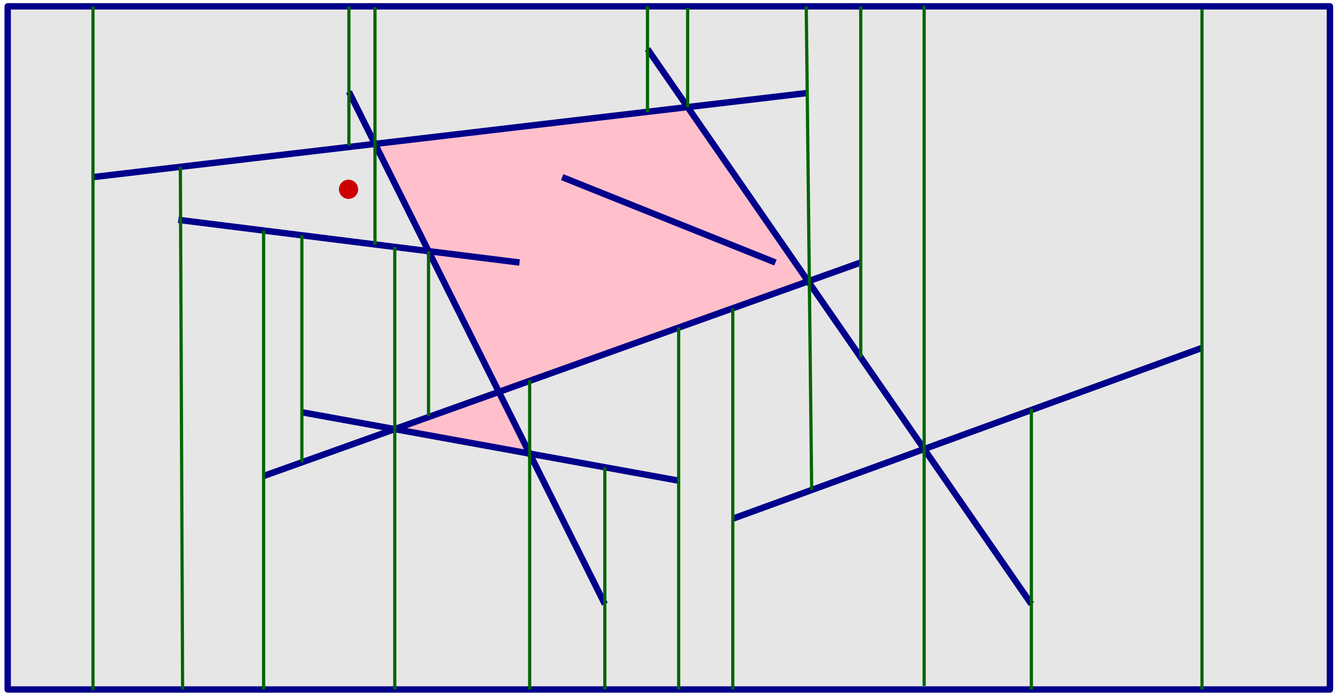


the 8-th iteration

clean-up phase:
remove trapezoids not in the cell of p

Theorem. Let S be a set of n line segments and let p be a point. Then the single cell of $\mathcal{A}(S)$ defined by p can be computed in $O(n\alpha(n)\log n)$ expected time.

- Apply standard RIC approach to construct trapezoidal decomposition of the whole arrangement.
- After iterations 1, 2, 4, 8, ... perform a clean-up step.

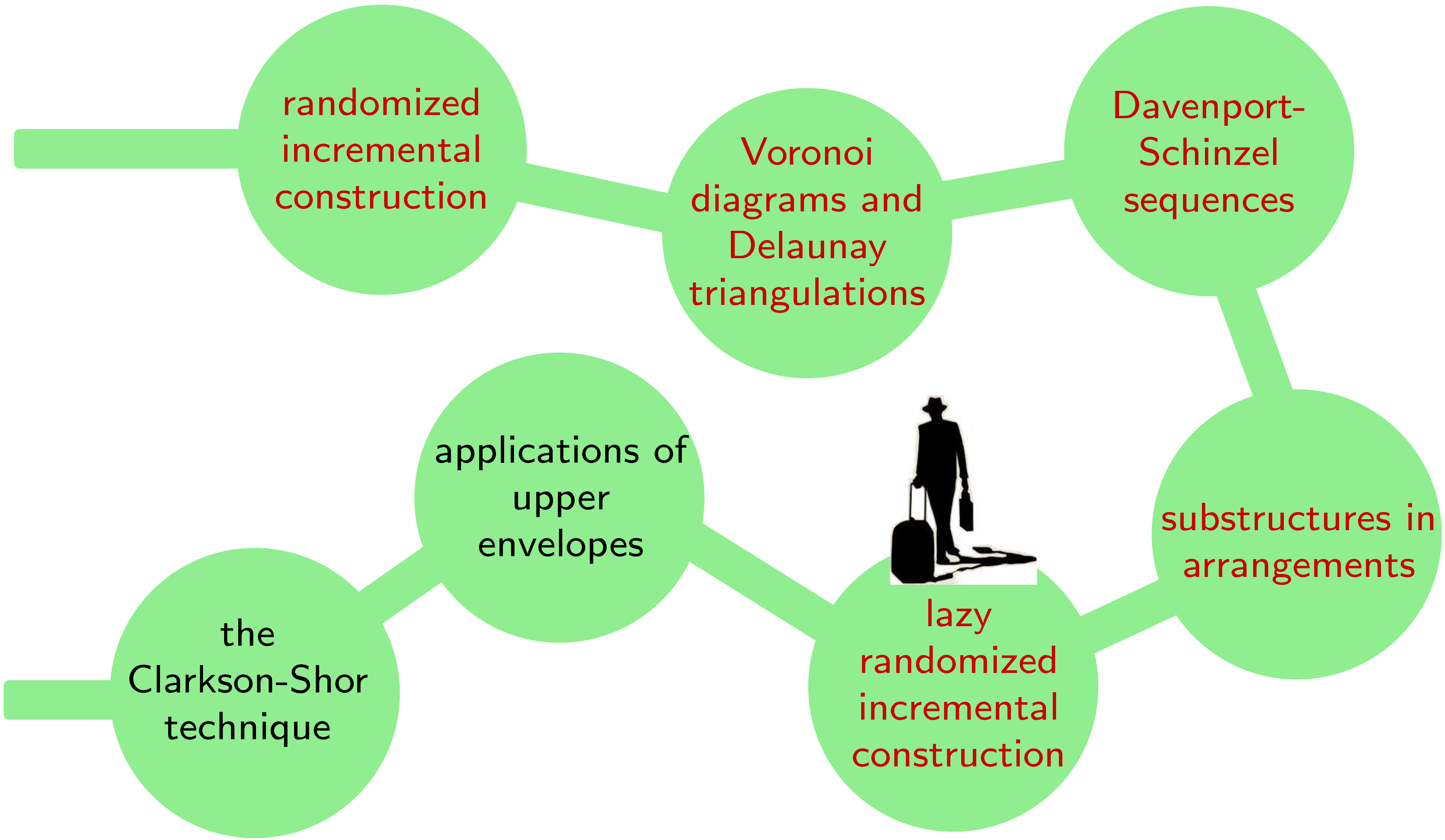


the 8-th iteration

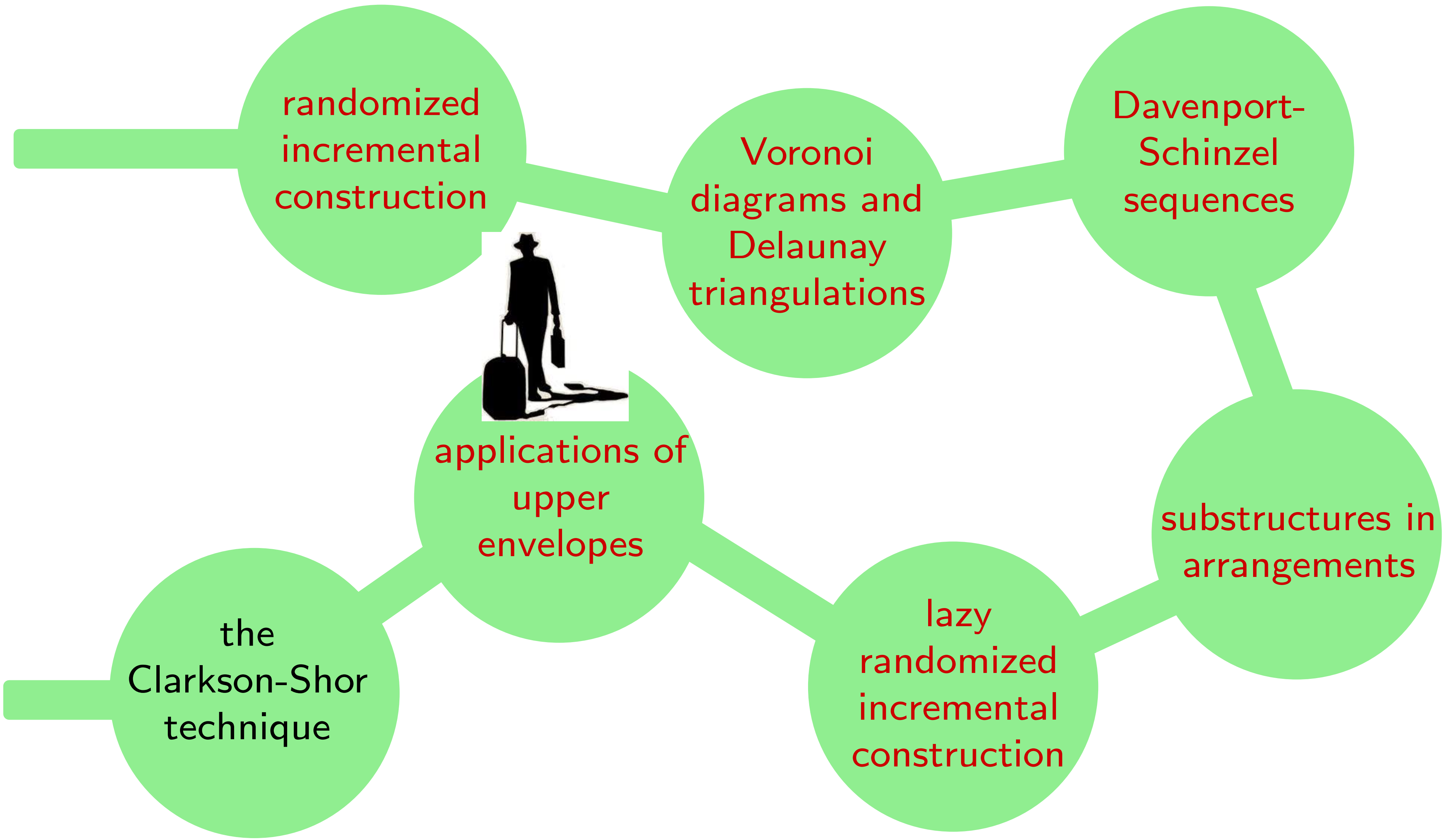
clean-up phase:
remove trapezoids not in the cell of p

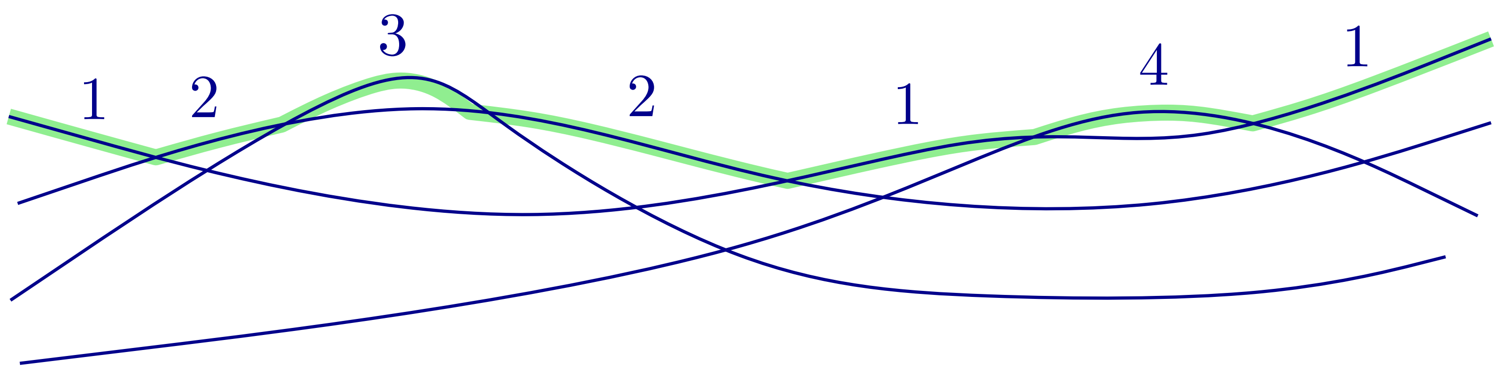
- Resulting algorithm has same performance bounds as when one could magically remove cells not in cell of p after each iteration
- Approach can also be formulated using abstract framework
- Can also be used to compute single cell in arrangement of triangles in $\mathbb{R}^3$, of zone of set of hyperplanes in $\mathbb{R}^d$, and more

Course Overview



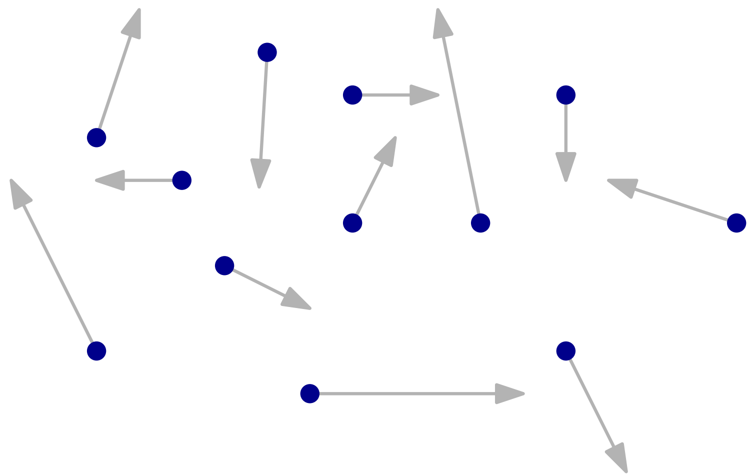
Course Overview





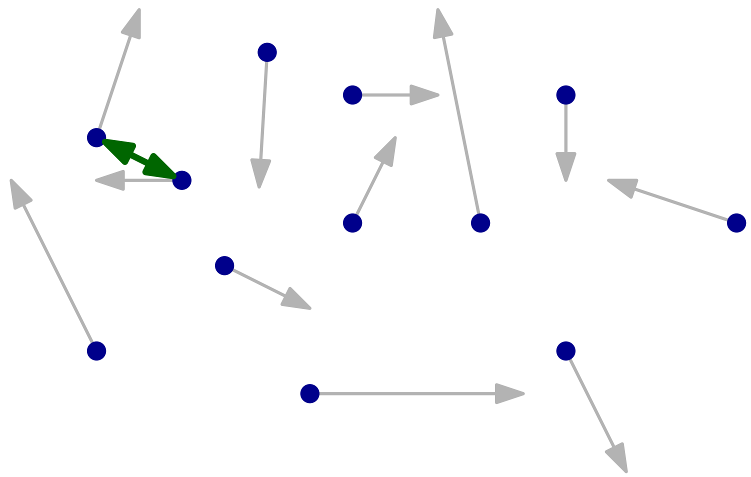
- n monotone curves with at most s intersections per pair
  - complexity of upper envelope is near-linear
  - infinite curves $O(\mathrm{DS}_s(n))$, finite curves $O(\mathrm{DS}_s(n))$
- n constant-degree algebraic surfaces in $\mathbb{R}^d$
  - complexity of upper envelope is $O(n^{d-1+\varepsilon})$, for any fixed $\varepsilon > 0$

P: set of n points in $\mathbb{R}^2$ that move linearly



- How often can the closest pair change, in the worst case?
- How often can the convex hull change, in the worst case?
- How often can the Delaunay triangulation change, in the worst case?

How often can the closest pair change, in the worst case?



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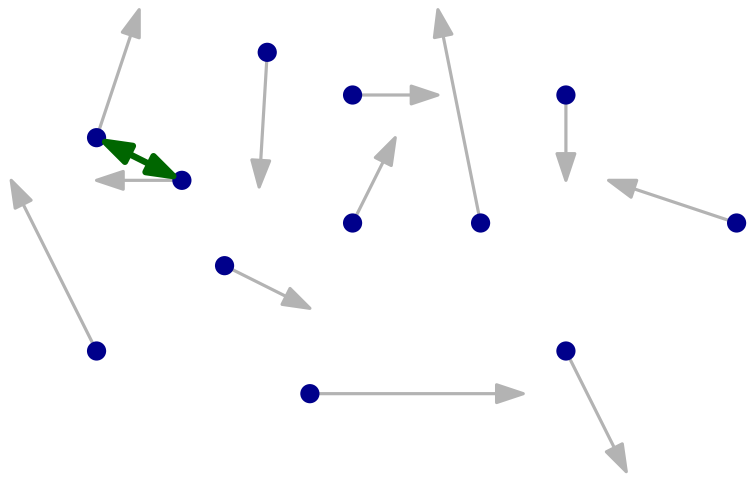
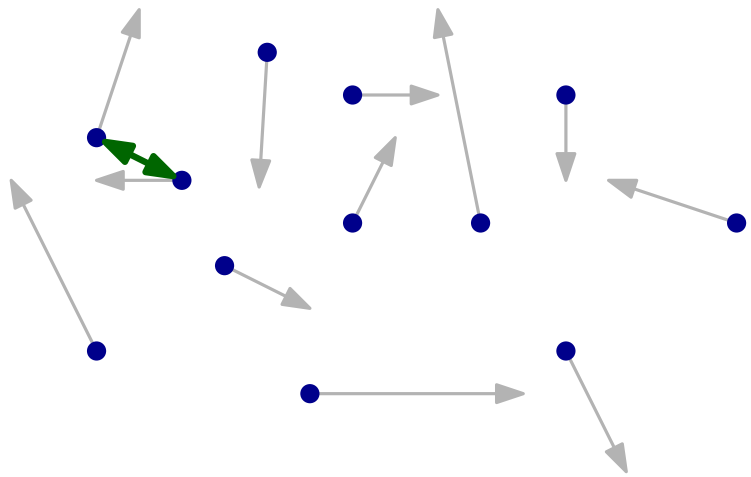


Lower bound

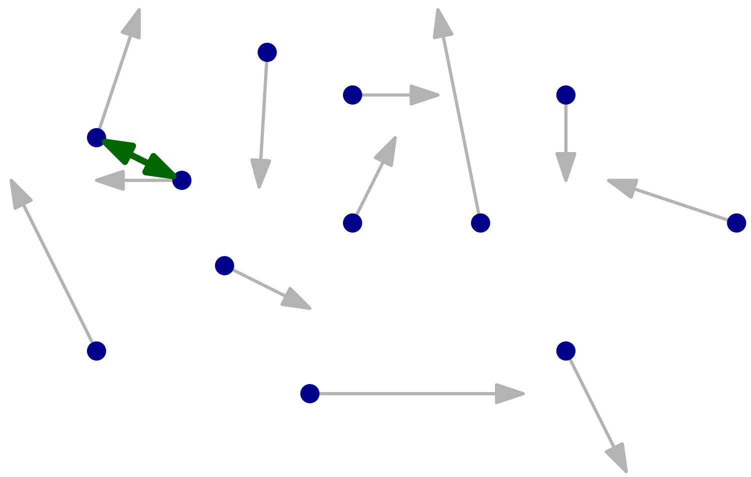
How often can the closest pair change, in the worst case?



Lower bound

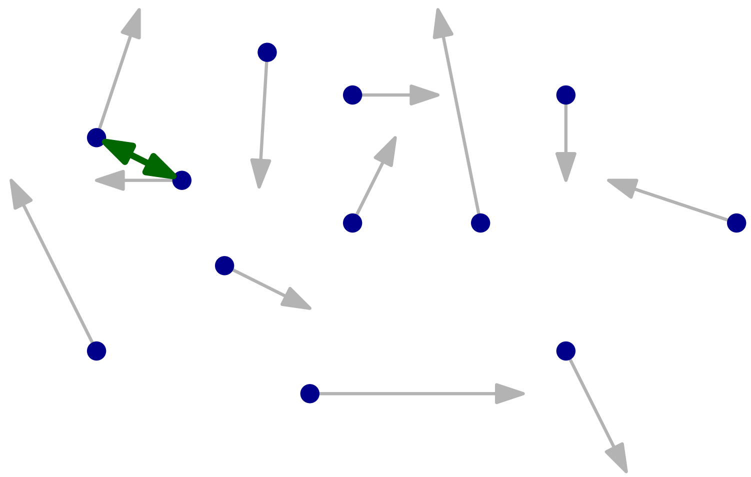
$\Omega(n^2)$ changes

How often can the closest pair change, in the worst case?



Upper bound

How often can the closest pair change, in the worst case?

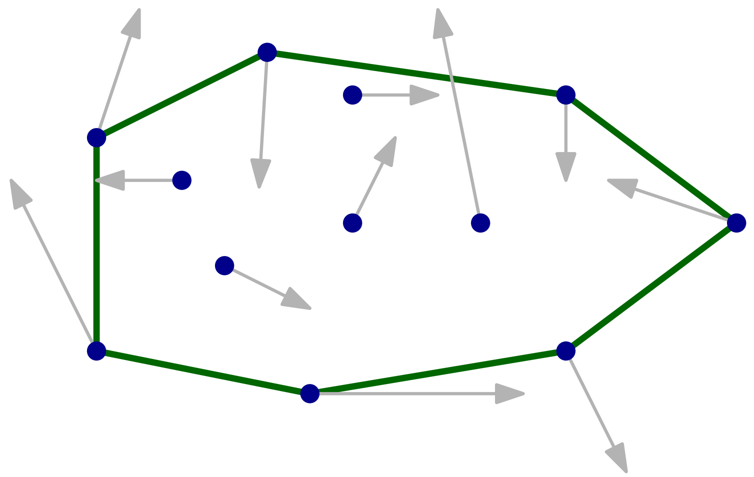


Upper bound

- for each pair p, q define $f_{pq}(t) :=$ distance between p and q at time t
- number of changes = complexity of lower envelope of n^2 functions

  $\approx O(n^2)$

How often can the convex hull change, in the worst case?



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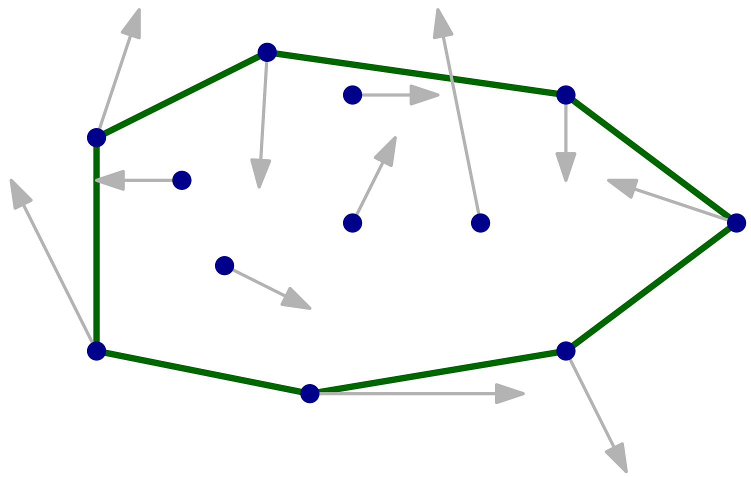
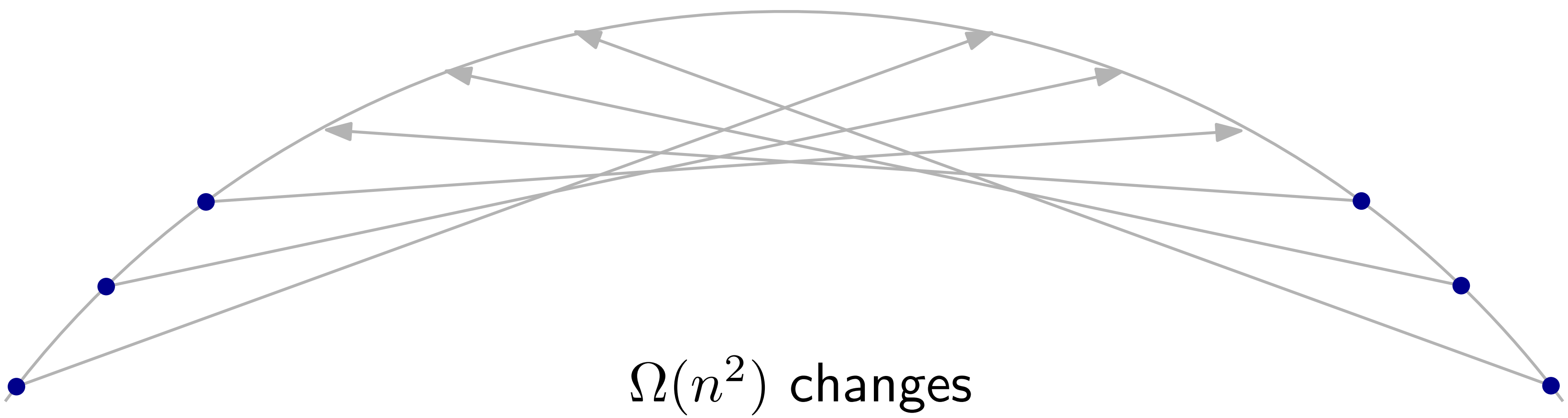


Lower bound



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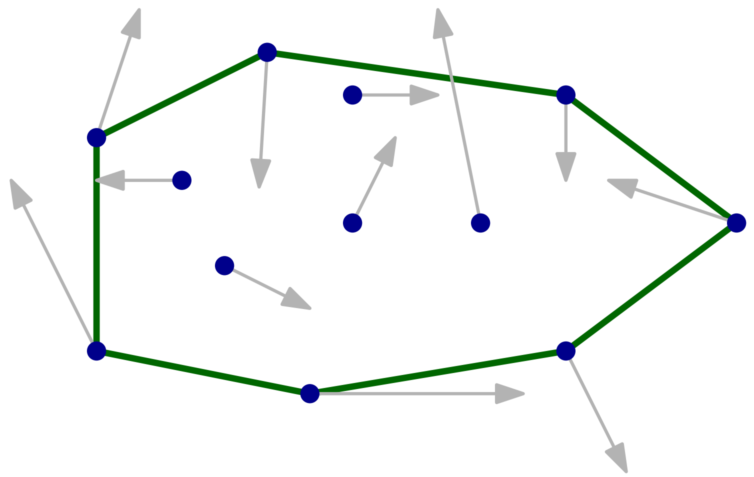

Trivial upper bound

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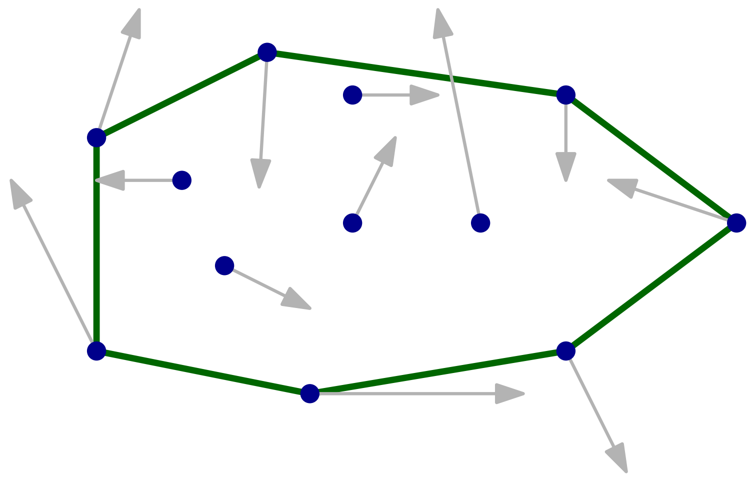


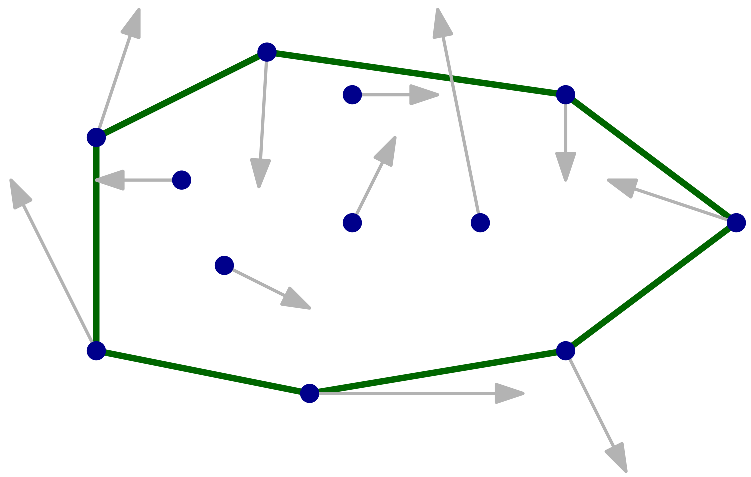
Trivial upper bound

convex hull changes $\Longrightarrow$ three points become collinear

$\Longrightarrow$ happens $O(1)$ times for each triple

$\Longrightarrow$ $O(n^3)$ changes to convex hull

How often can the convex hull change, in the worst case?



A better bound using upper envelopes

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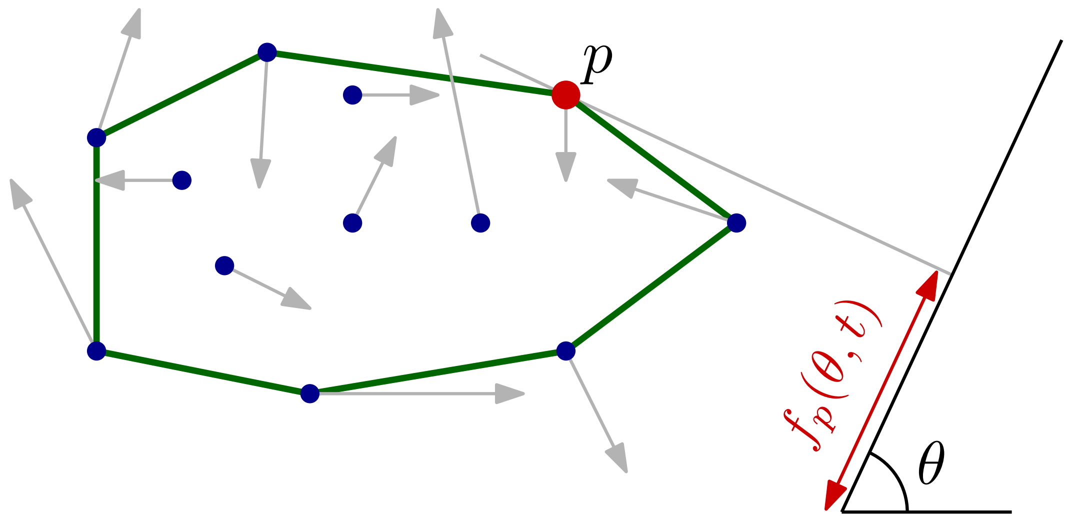

A better bound using upper envelopes

- for each point p define function $f_p : [0, 2\pi) \times \mathbb{R}_{\geqslant 0} \to \mathbb{R}$

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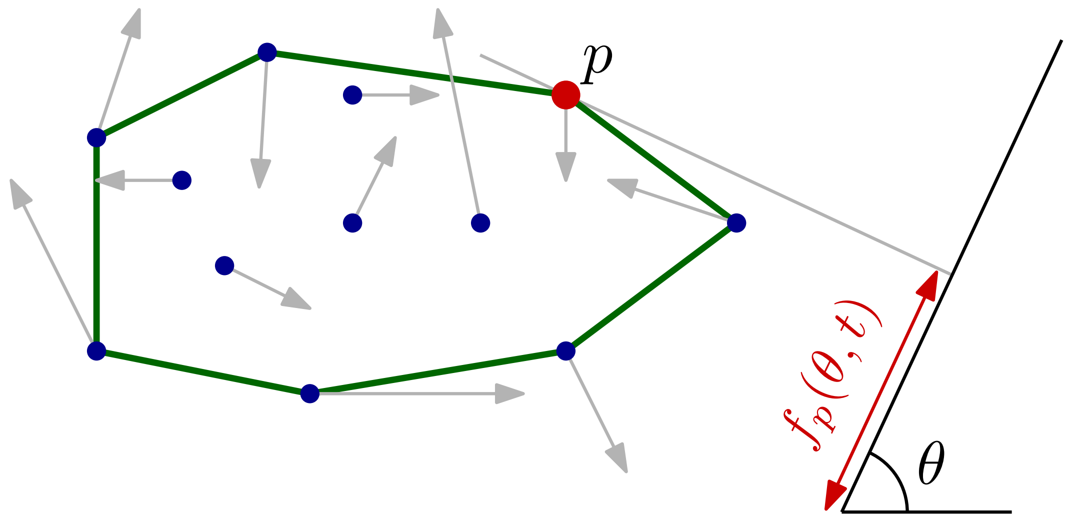

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- for each point p define function $f_p : [0, 2\pi) \times \mathbb{R}_{\geqslant 0} \to \mathbb{R}$
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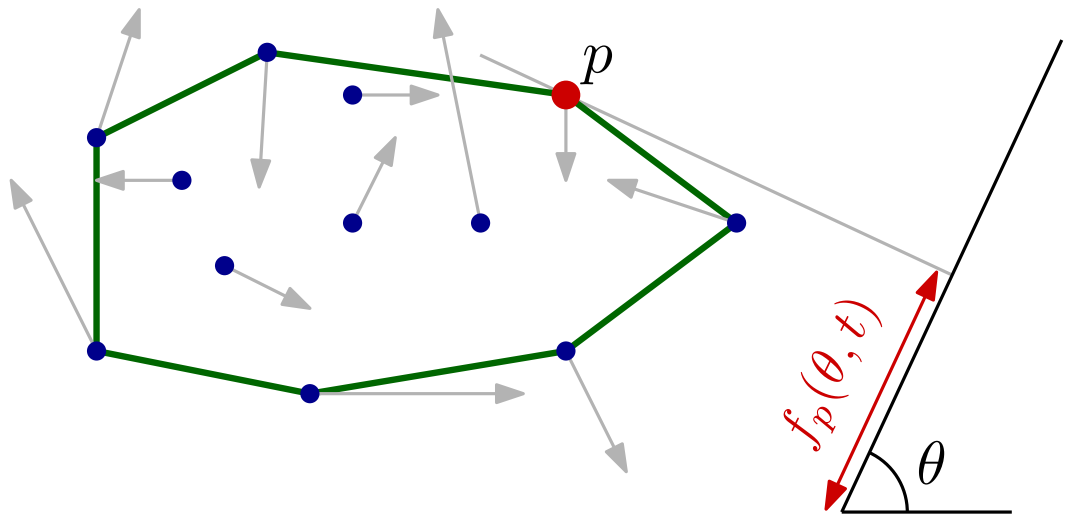
How often can the convex hull change, in the worst case?



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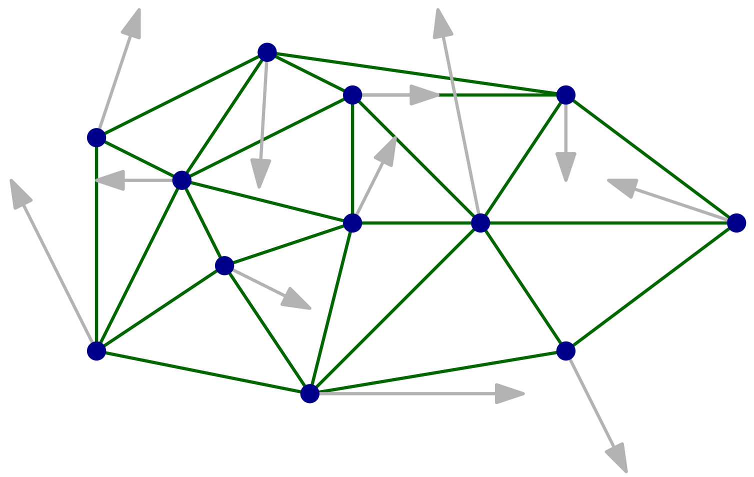
- p is on convex hull at time t iff $f_p(\theta, t) \geqslant f_q(\theta, t)$ for all q at time t
- number of changes
  $= O($complexity of upper envelope of surfaces in $\mathbb{R}^3) = O(n^{2+\varepsilon})$

How often can the Delaunay triangulation change, in the worst case?



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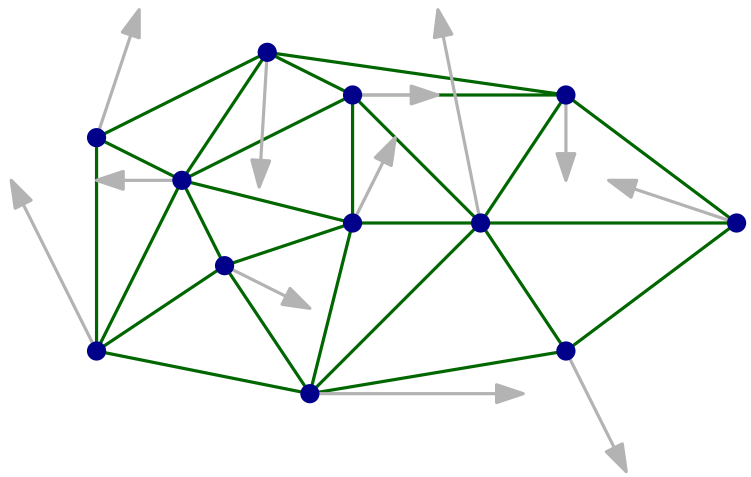
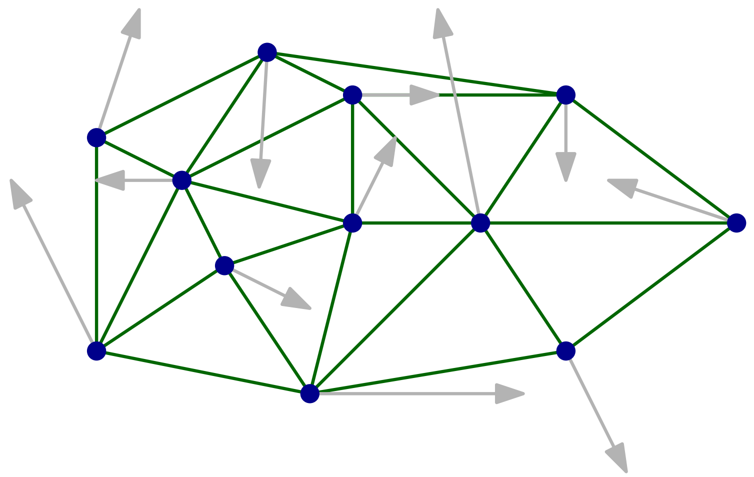


DT changes when convex hull changes $\Longrightarrow \Omega(n^2)$ changes

Exercises

1. Give a trivial upper bound on the number of changes.
2. Give an improved upper bound using upper envelopes.

How often can the Delaunay triangulation change, in the worst case?



1. When DT changes, four points become co-circular $\implies$ $O(n^4)$ changes

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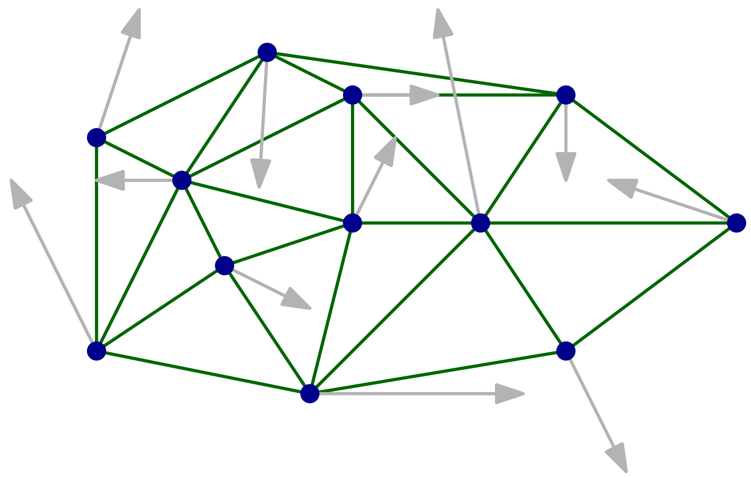
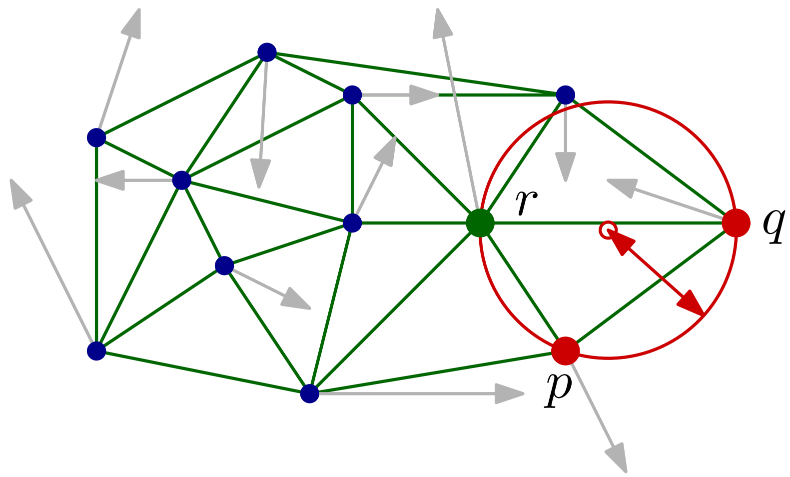


1. When DT changes, four points become co-circular $\implies$ $O(n^4)$ changes
2. When convex hull changes, DT changes $\implies$ $\Omega(n^2)$ changes

How often can the Delaunay triangulation change, in the worst case?



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2. for each pair p, q, and each r, define function $f_{pq}^{(r)}(t) : \mathbb{R}_{\geqslant 0} \to \mathbb{R}$

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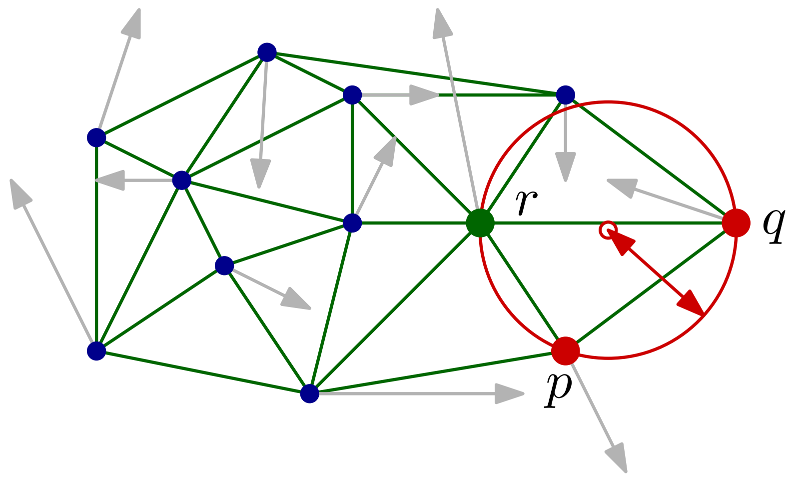

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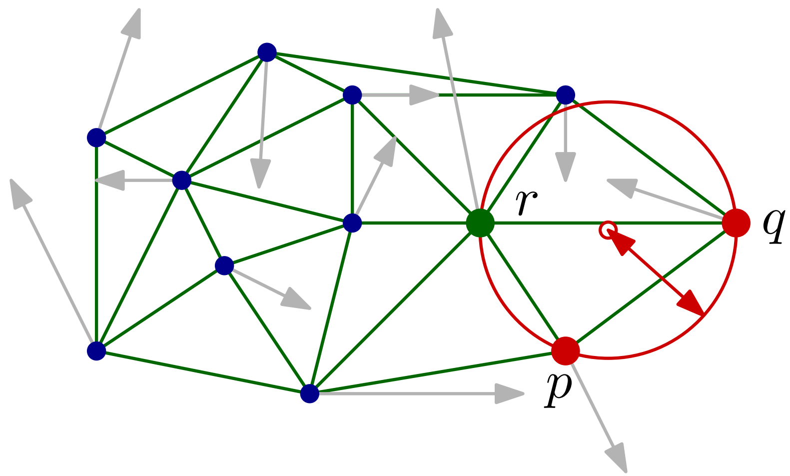

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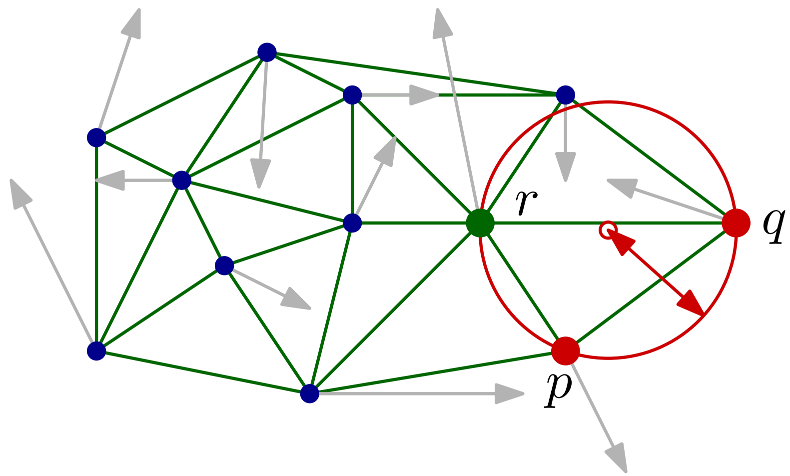

[Rubin '15; 85 pages]
for linear motions the DT changes $O(n^{2+\varepsilon})$ times

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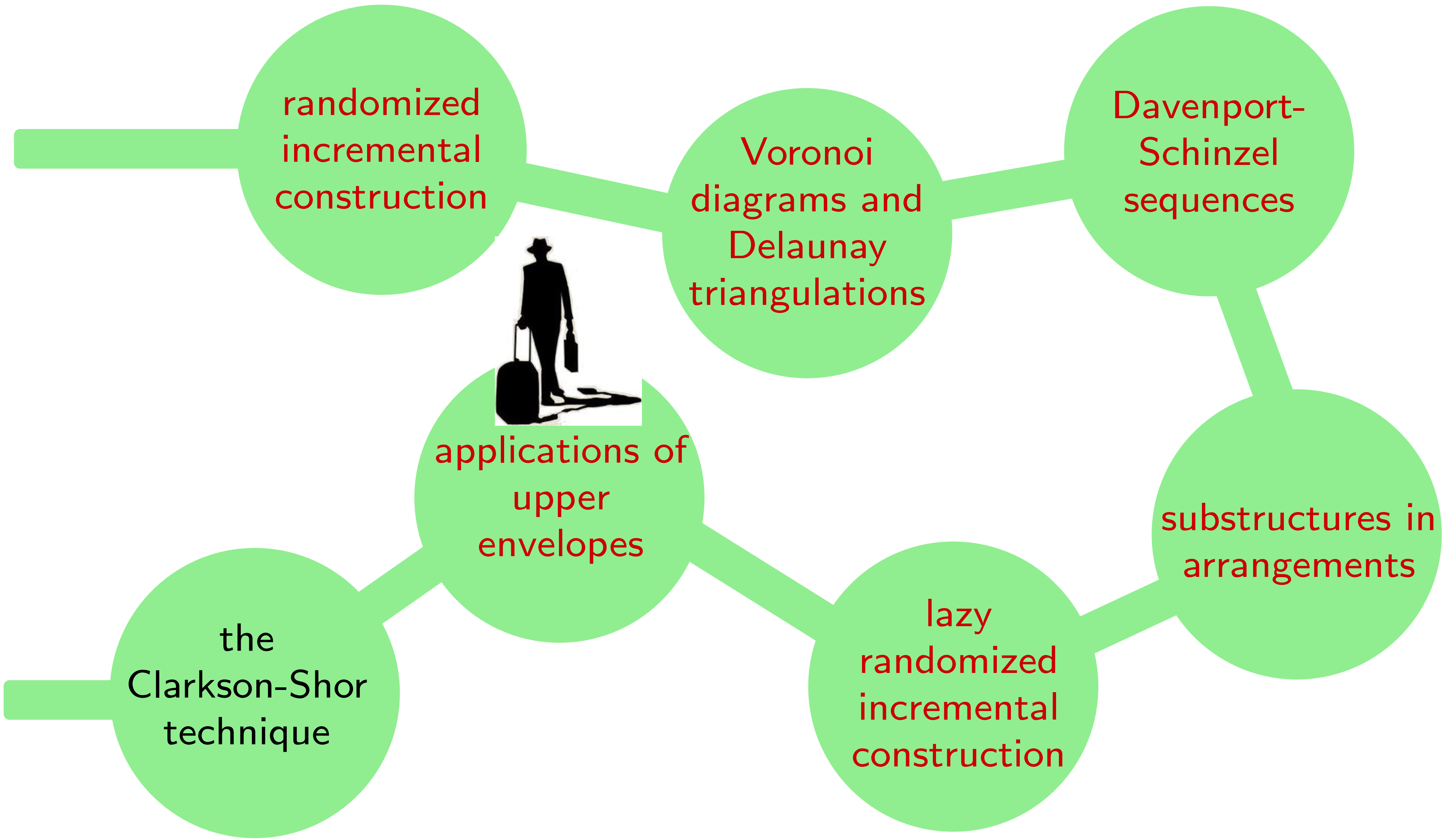
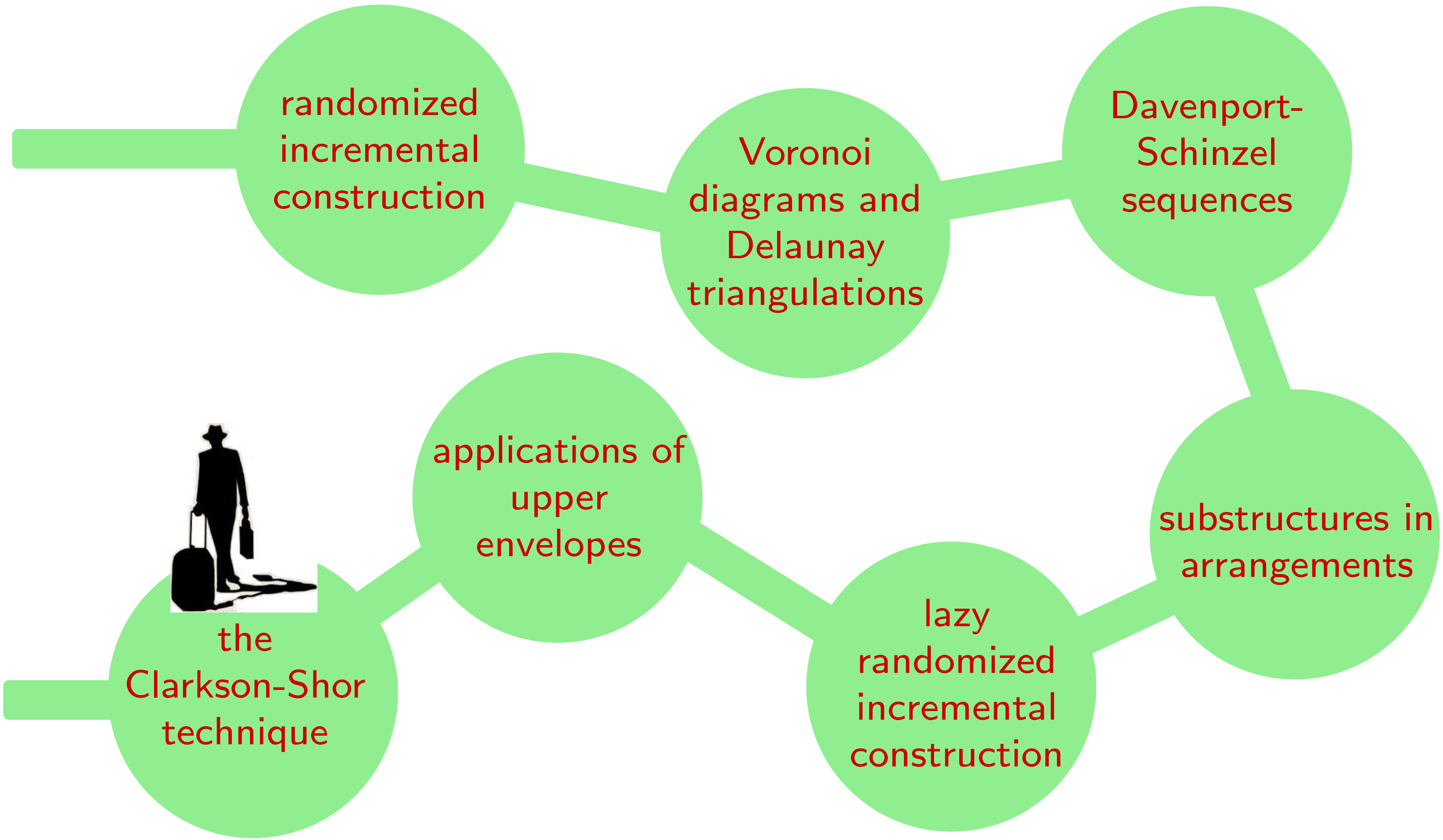
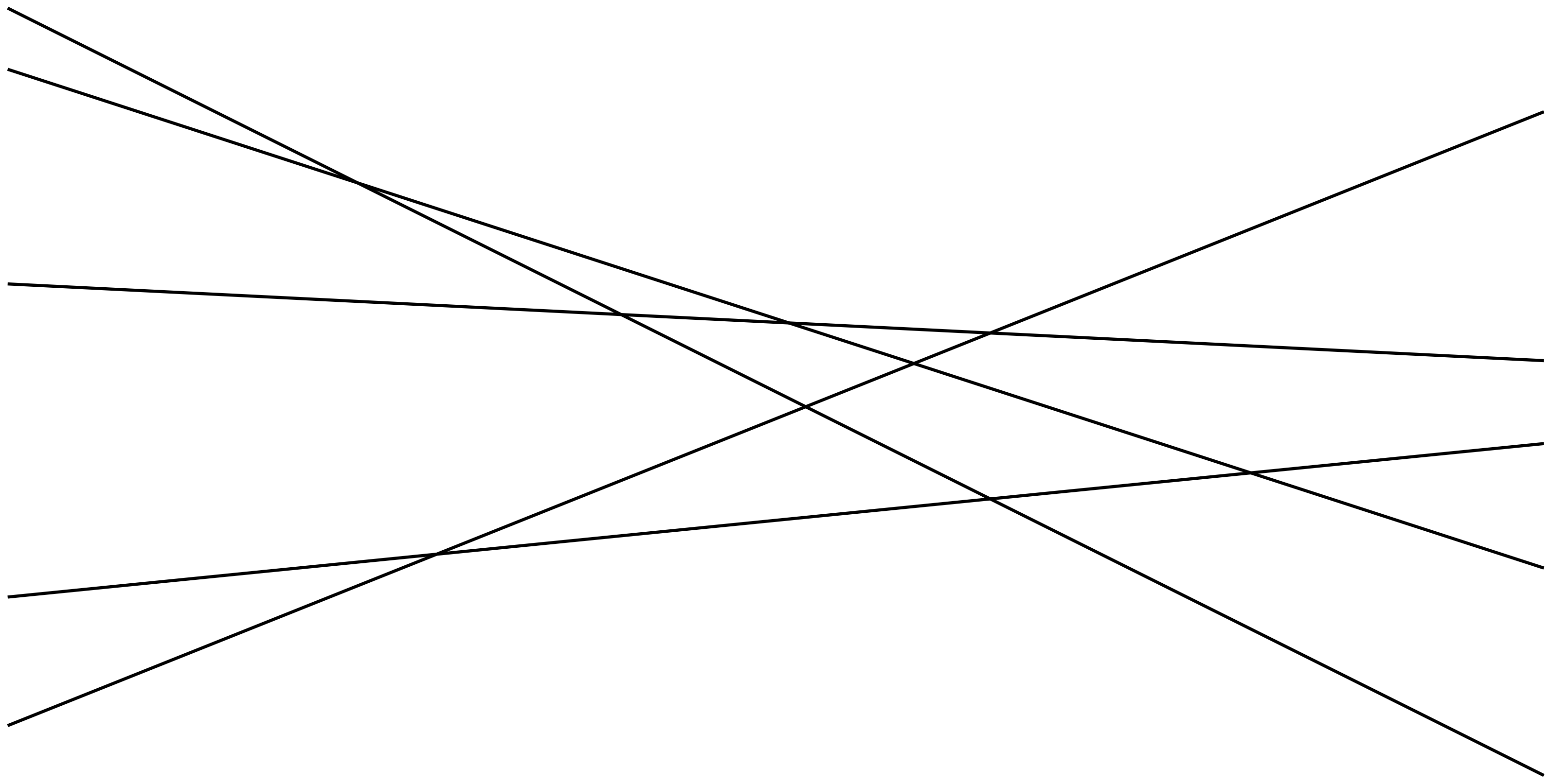
Course Overview

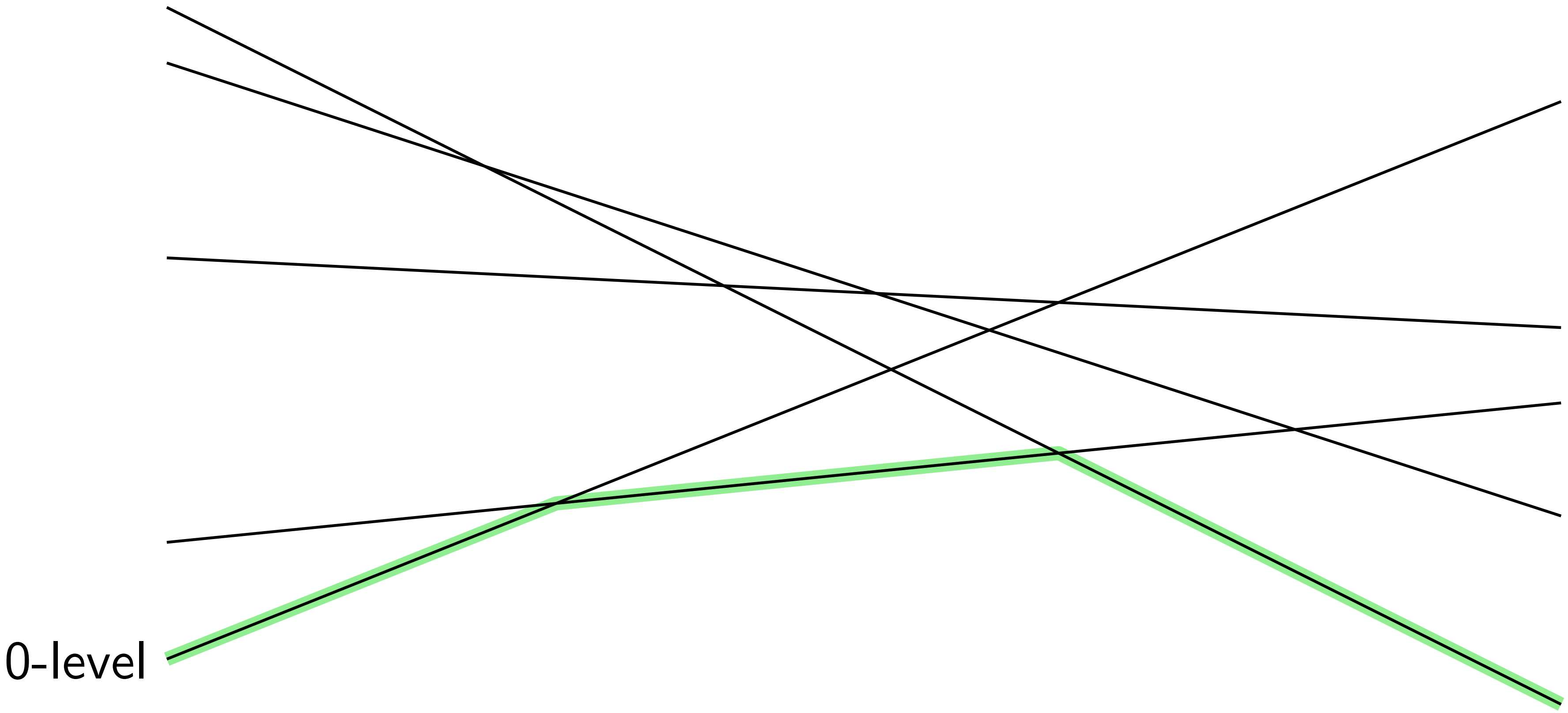


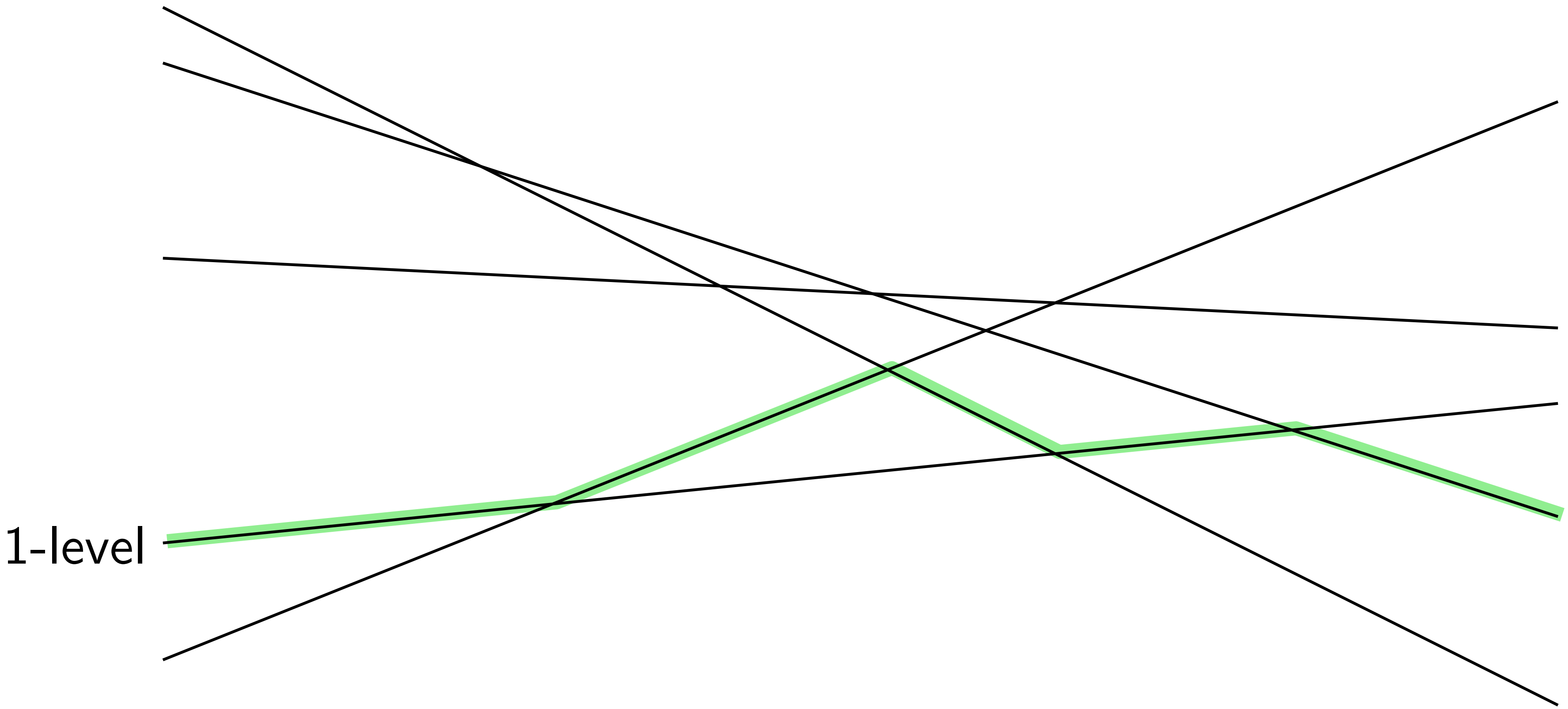
Course Overview

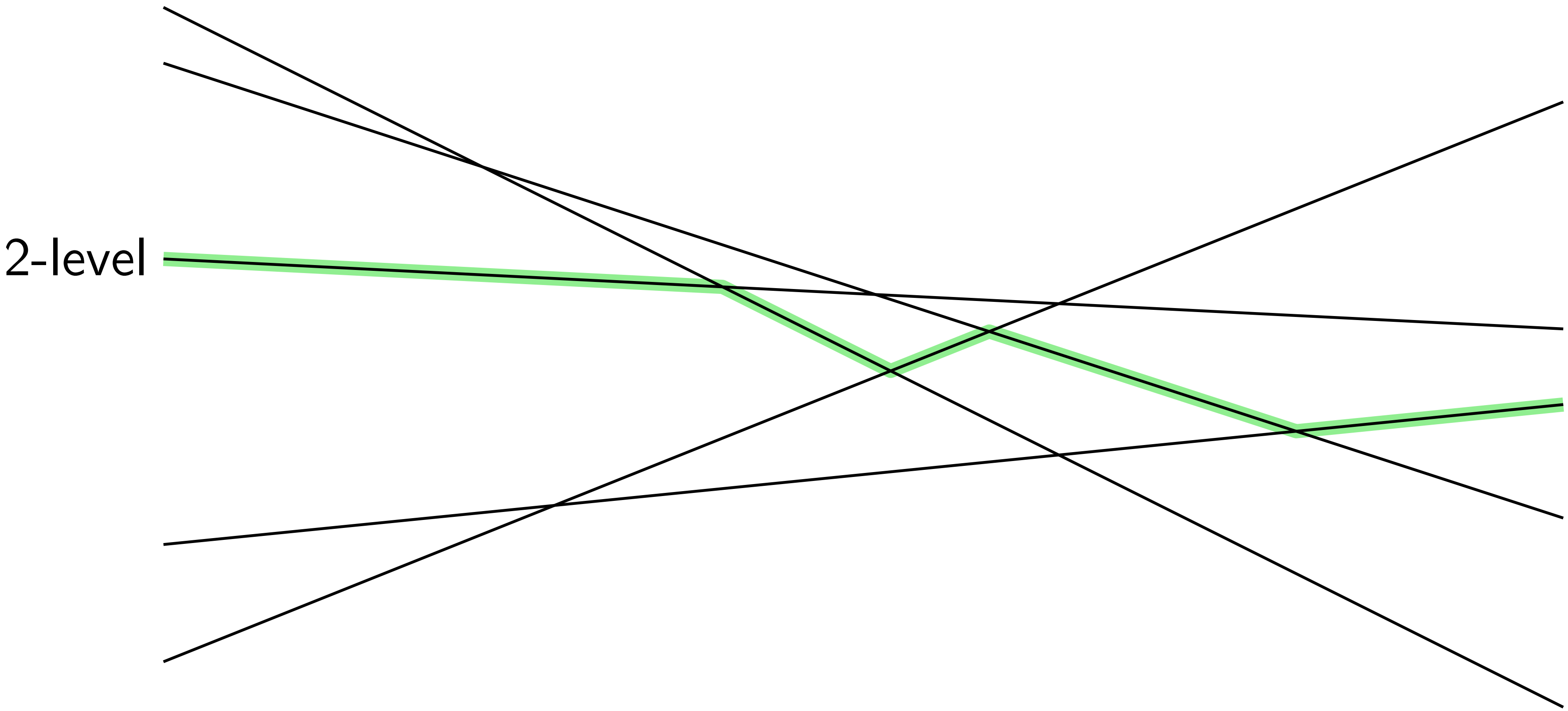


Levels in arrangements

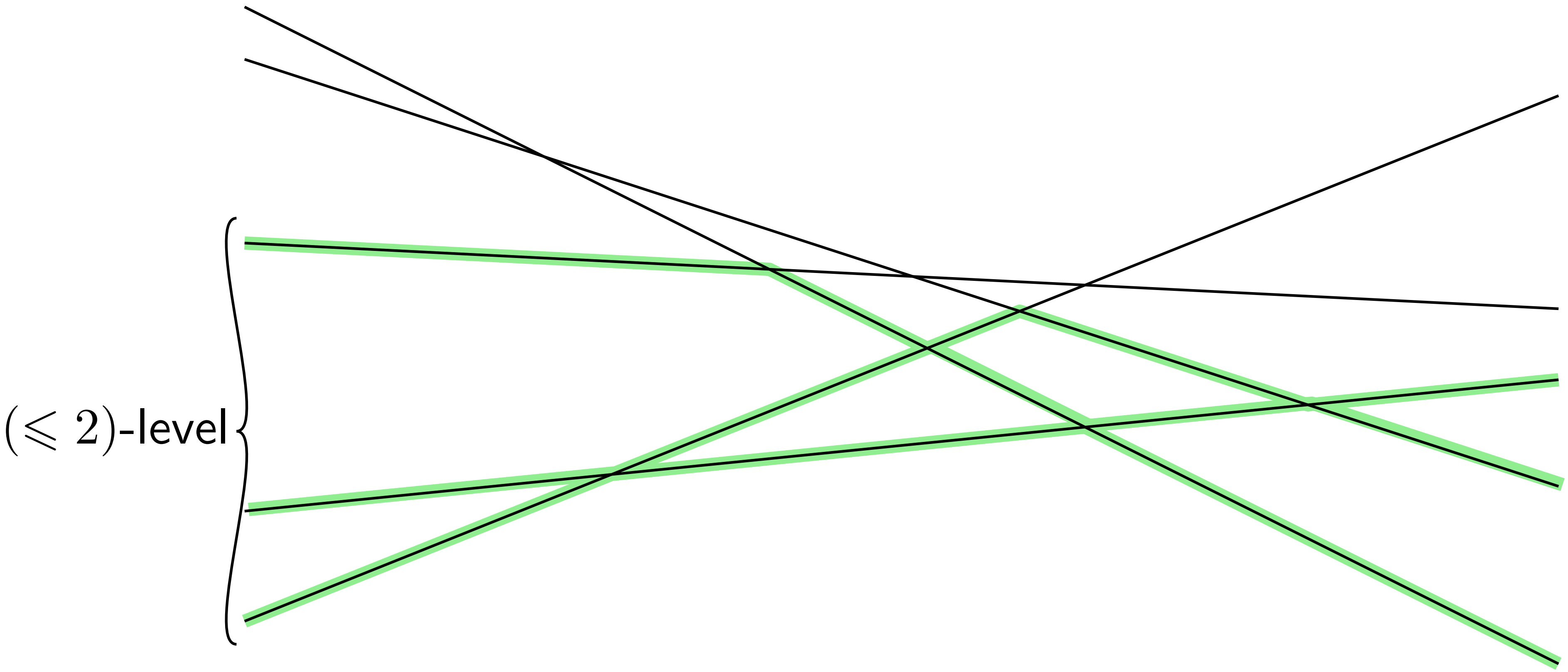



0-level

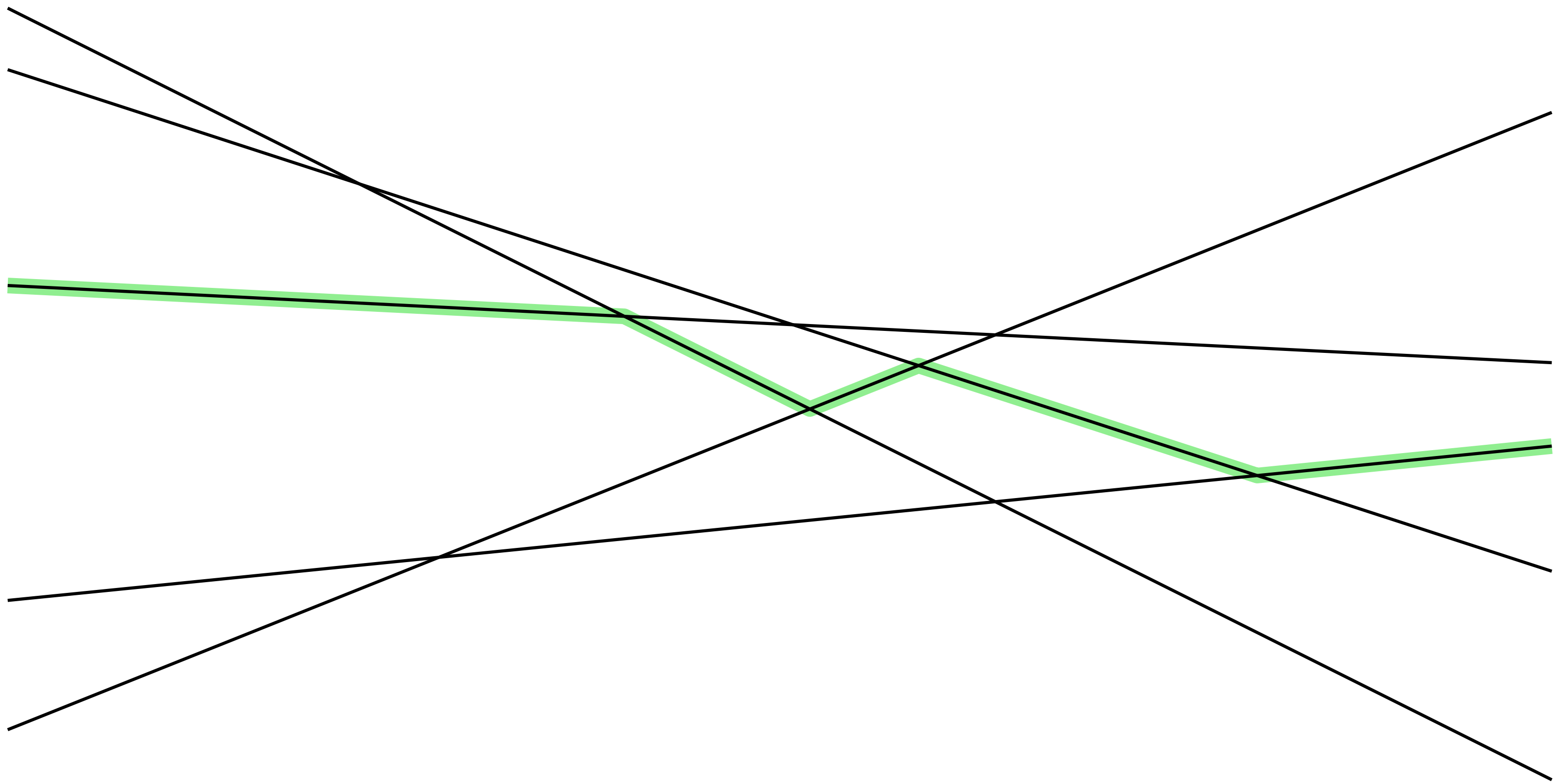

1-level


2-level

Levels in arrangements



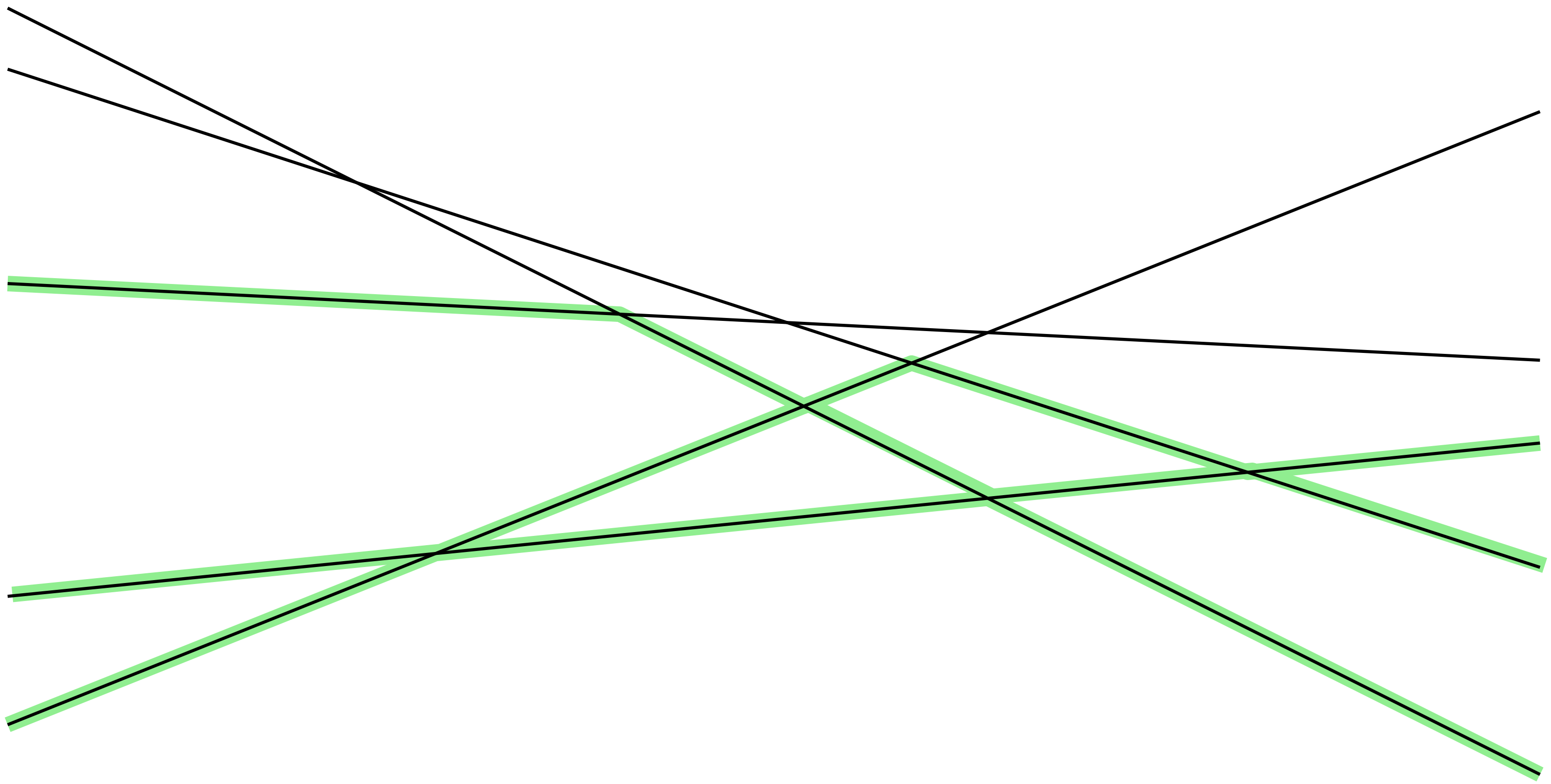
Levels in arrangements



What is the max complexity of the k-level in an arrangement of n lines?

- 0-level $=$ lower envelope $\implies$ complexity $\leqslant n$
- $k \geqslant 1$: complexity is $n2^{\Omega(\sqrt{\log k})}$ and $O(nk^{1/3})$

The Clarkson-Shor Technique: Application to $(\leqslant k)$-levels



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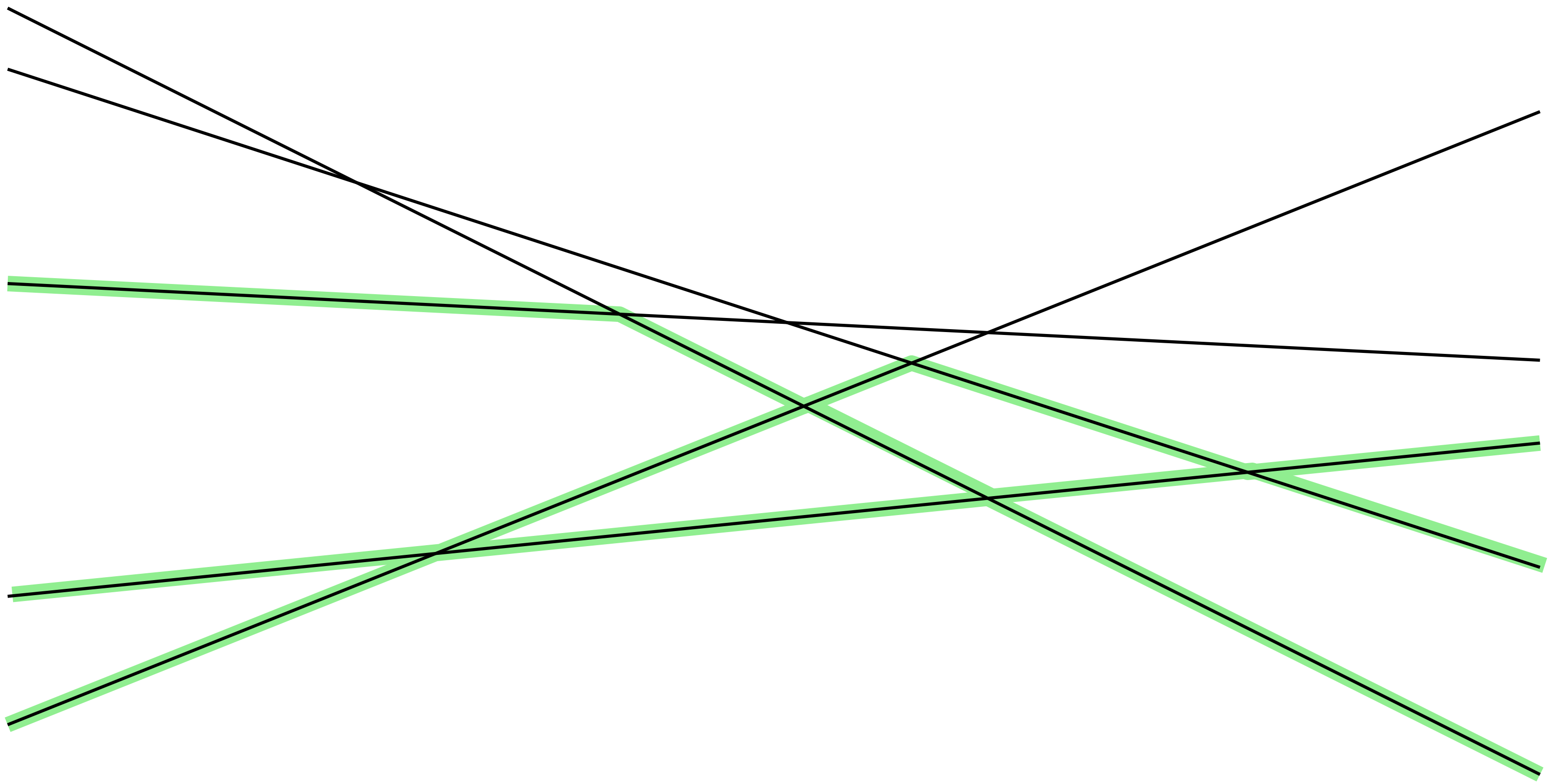
The Clarkson-Shor Technique: Application to $(\leqslant k)$-levels



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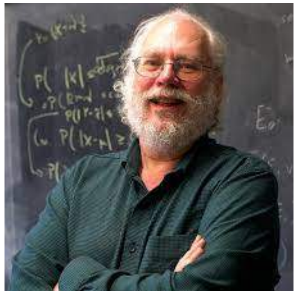

Clarkson-Shor '89: $\Theta(nk)$

The Clarkson-Shor Technique: Application to $(\leqslant k)$-levels

Theorem. The max complexity of the $(\leqslant k)$-level in an arrangement induced by a set L of n lines in the plane is $O(nk)$.

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Proof.

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Take sample $R \subset L$ by picking each line $\ell \in L$ with probability $1/k$.

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$$\mathbb{E}\big[\text{complexity of 0-level of } R\big] \;\leqslant\; \mathbb{E}\big[|R|\big] \;=\; n/k$$

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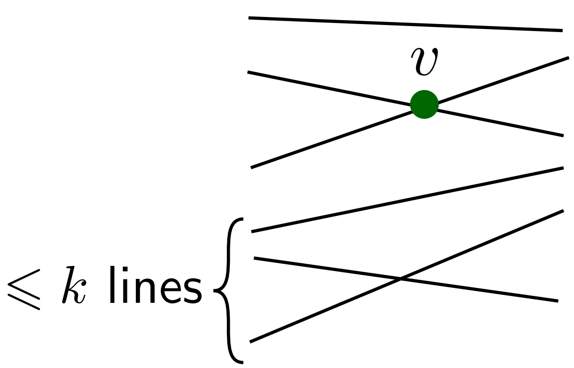

vertex of k-level of L shows up on 0-level of R iff

- both lines defining v are in R
- none of the at most k lines below v are in R

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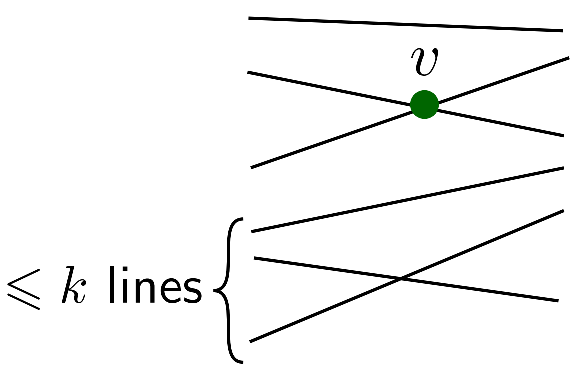

vertex of k-level of L shows up on 0-level of R iff

- both lines defining v are in R
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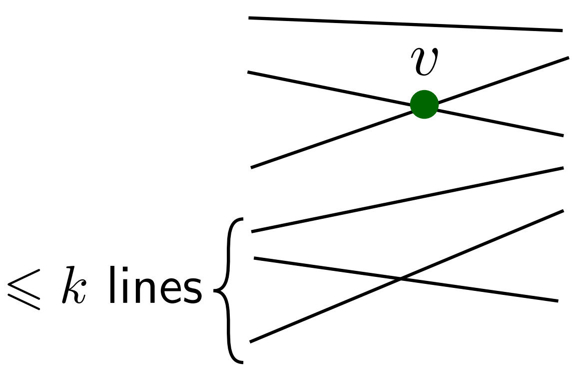
$$\text{prob} \geqslant \left(\tfrac{1}{k}\right)^2 \cdot \left(1 - \tfrac{1}{k}\right)^k \geqslant \left(\tfrac{1}{k}\right)^2 \cdot \tfrac{1}{e}$$

Theorem. The max complexity of the $(\leqslant k)$-level in an arrangement induced by a set L of n lines in the plane is $O(nk)$.

Proof.

Take sample $R \subset L$ by picking each line $\ell \in L$ with probability $1/k$.

$$\mathbb{E}\big[\text{complexity of 0-level of } R\big] \;\leqslant\; \mathbb{E}\big[|R|\big] \;=\; n/k$$



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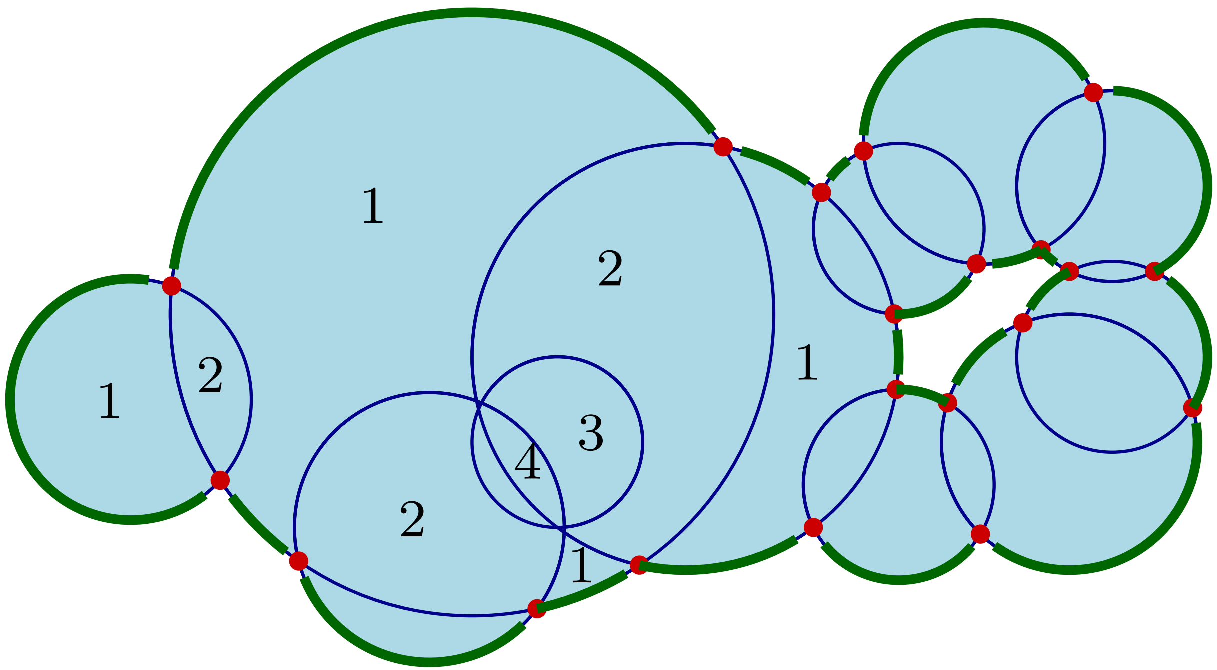
$$\mathbb{E}\big[\text{complexity of 0-level of } R\big] \;\geqslant\; (\text{complexity of } k\text{-level in } L) \cdot \left(\tfrac{1}{k}\right)^2 \cdot \tfrac{1}{e}$$

Another application: Depth in Disk Arrangements

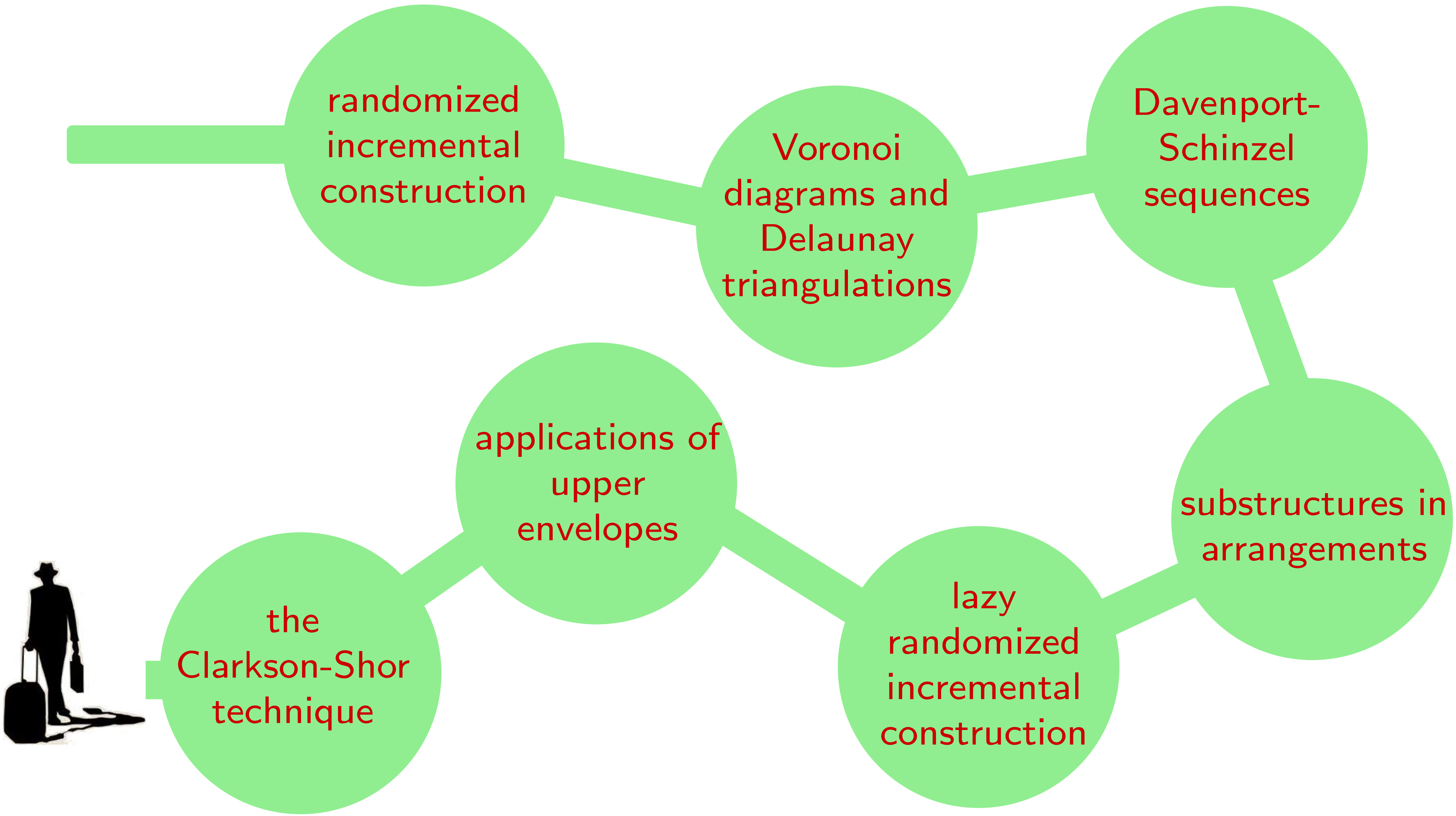


Exercises

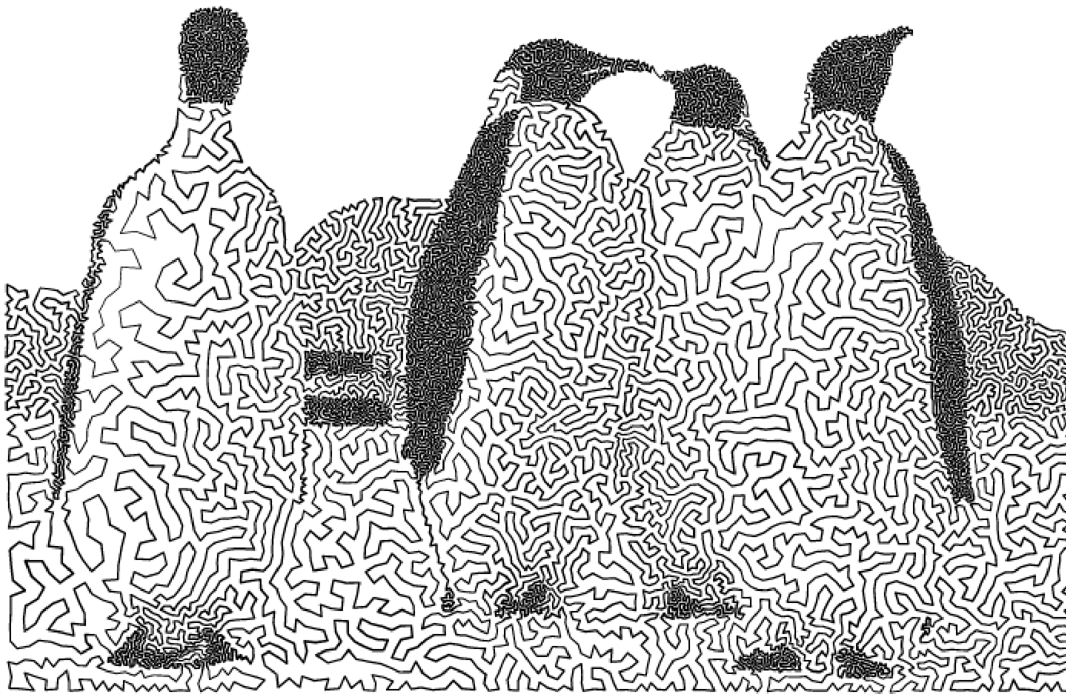
1. Prove that the total number of vertices on the union boundary is $O(n)$. Hint: Define a suitable planar graph whose nodes are disk centers.

2. Prove that the total number of regions of depth at most k is $O(nk)$.

Course Overview



Thanks for your attention!



TSP Art by Carig Kaplan and Robert Bosch